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THE PLASTIC WRINKLING OF AN ANNULAR PLATE UNDER UNIFORM TENSION ON ITS INNER EDGE

L. C. ZHANG and T. X. YU
Department of Mechanics, Peking University, People's Republic of China

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Abstract—This paper analyses the plastic wrinkling of an annular plate subjected to in-plane uniform tension stress on its inner edge with the combined use of the Kantorovich method and the Galerkin method, and discusses the appearance of wrinkles on the flange of a metal circular sheet during its axisymmetric deep-drawing operation. It is shown that the method provided in this paper is simple, convenient, and very suitable for engineering applications.

NOTATION

a	inner radius of an annular plate
b	outer radius of an annular plate
c	undetermined coefficient in approximate wrinkling mode
$D_{\#}$	parameter defined by eqn (17)
E	Young's modulus
E_s, E_s^0	secant moduli of effective stress–strain curve and uniaxial stress–strain curve, respectively
e_{ij}	deviatoric strain components
f	function of non-dimensional radial coordinate ρ
g	function of circumferential coordinate θ
L_1, L_2	differential operators
n	number of waves
r, θ	polar coordinates
s_{ij}	deviatoric stress components
t	plate thickness
w	mode of wrinkling
Y	yield stress of the plate material
ε_{ij}	strain components
$\bar{\varepsilon}$	effective strain
ζ	non-dimensional parameter defined by eqn (29)
$\lambda_1, \lambda_2, \lambda_3$	parameters defined by eqns (26)
ν	Poisson's ratio in the elastic regime
$\nu_{\#}$	parameter defined by eqn (19)
ς	non-dimensional parameter, $1-\rho_{\#}$
$\rho, \rho_{\#}$	non-dimensional radial coordinate, r/b, and one of its values, a/b
$\bar{\sigma}$	effective stress
σ_{ij}	stress tensor
ϕ, ψ	parameters defined by eqns (3a) and (3b), respectively.

1. INTRODUCTION

When the inner edge of an annular plate is loaded by an in-plane uniform tensile stress, the stress state of the plate, according to the theory of plasticity, can be expressed as

$$\begin{cases} \sigma_r = Y \ln (b/r) \\ \sigma_\theta = Y[\ln (b/r) - 1] \\ \sigma_{r\theta} = 0 \end{cases} \tag{1}$$

if the plate material is regarded as being perfectly plastic and the total plate has yielded. As

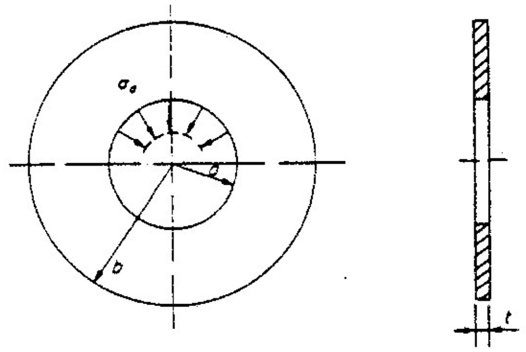

Fig. 1. An annular plate under in-plane tension.

the circumferential stress, σ_θ, is compressive (Fig. 1), plastic buckling will occur circumferentially, when the uniform boundary stress, σ_a, reaches a critical value. This appearance is called *plastic wrinkling*.

Investigation of wrinkling has a great deal of significance in many branches of mechanical engineering, especially in the axisymmetric deep-drawing process of circular metal sheets (Fig. 2). Engineers require that the flange of a workpiece in its deep-drawing operation should deform in its plane and not wrinkle because otherwise it will impair the quality of the product. For this reason the wrinkling of an annular plate has been the focus of much research attention over several decades. Geckeler[1] simplified the problem and treated it as a one-dimensional model, furnishing some formulae to predict the critical circumferential stress and the number of waves. His model was later employed and extended by many other researchers[2–4]. Reference [5] realized the limitation of Geckeler's one-dimensional model and studied the problem using a two-dimensional one by means of the energy method. However, as recently pointed out by the authors[7], their results may still be too simplistic for general application.

In this paper, the plastic wrinkling equation is derived based on the aspect of stability according to the two-dimensional model for the wrinkling analysis of an annular plate in Ref. [5], and then the title problem is solved by the combined use of the methods of Kantorovich and Galerkin[8]. The criterion for predicting plastic wrinkling is obtained and the applications of the present results to the deep-drawing operation in sheet forming are discussed in detail. Some useful conclusions are obtained. It is shown that the method provided in this paper is simple, convenient and very suitable for engineering applications.

2. SOLUTION

2.1. *Plastic wrinkling equation*

According to the theory of plasticity[5], we have

$$\begin{cases} \delta e_{ij} = \psi \delta s_{ij} \\ \delta \varepsilon_{kk} = \phi \delta \sigma_{kk} \end{cases} \tag{2}$$

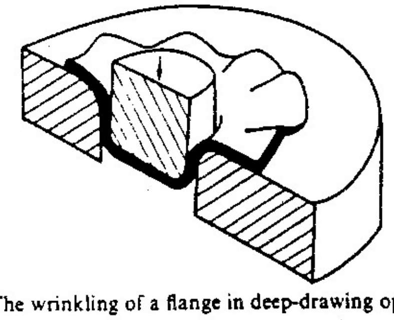

Fig. 2. The wrinkling of a flange in deep-drawing operation.

where e_{ij} and s_{ij} are deviatoric strain and deviatoric stress tensor, and ε_{ij} and σ_{ij} are strain and stress components, respectively. The notation "$\delta(x)$" indicates an infinitesimal increment of a corresponding physical variable. Parameters ϕ and ψ are defined by

$$\phi = (1-2\nu)\,E \tag{3a}$$

and

$$\psi = 3\bar{\varepsilon}\,(2\bar{\sigma}) \tag{3b}$$

where E is Young's modulus and ν Poisson's ratio. $\bar{\varepsilon}$ and $\bar{\sigma}$ are effective strain and effective stress, and are defined by

$$\bar{\varepsilon} = (\tfrac{2}{3}e_{ij}e_{ij})^{1/2}$$
$$\bar{\sigma} = (\tfrac{3}{2}s_{ij}s_{ij})^{1/2}$$

respectively. By introducing the secant modulus

$$E_s = \bar{\sigma}/\bar{\varepsilon} \tag{4}$$

the following expression can be obtained from the uniaxial stress–strain state:

$$\frac{1}{E_s^0} = \frac{1}{E_s} + \frac{1-2\nu}{3E} \tag{5}$$

where E_s^0 is the secant modulus of the uniaxial stress–strain curve. In this way, we get

$$\frac{\bar{\sigma}}{\bar{\varepsilon}} = \frac{3E^0}{3-\phi E_s^0}. \tag{6}$$

For the annular plate shown in Fig. 1, wrinkling always occurs first at the outer edge because $|\sigma_\theta|_{r=b}$ is maximum through the plate, according to eqns (1). Therefore, the following selection is wise and convenient for the wrinkling prediction of the annular plate, that is

$$E_s^0 = E_s^0|_{r=b}. \tag{7}$$

Keeping these in mind and noting that stresses are uniformly distributed through the plate thickness before wrinkling occurs so that E_s is independent of the coordinate z, the wrinkling differential equation for the annular plate can be expressed as

$$D_p\nabla_*^4 w - \left[N_r\frac{\partial^2 w}{\partial r^2} + N_\theta\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}\right)\right] = 0 \tag{8}$$

where

$$N_r = t\sigma_r,\quad N_\theta = t\sigma_\theta \tag{9}$$

$$\nabla_*^4 = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)^2 \tag{10}$$

$$D_p = t^3\,[6(1+\phi E_s^0)\psi]. \tag{11}$$

Obviously, when $E_s^0 = E$, eqn (8) reduces to the elastic wrinkling equation.

Substituting eqns (1) into eqn (8), and introducing the notation

$$\rho = r/b \tag{12}$$

the non-dimensional plastic wrinkling equation of the annular plate can be written as

$$D_{\#}\nabla^4 w + \ln(\rho)\nabla^2 w + \Box w = 0 \tag{13}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2} \tag{14}$$

$$\nabla^4 = \nabla^2\nabla^2 \tag{15}$$

$$\Box = \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2} \tag{16}$$

and

$$D_{\#} = t^2/[6Yb^2(1+\phi E_s^0)\psi]. \tag{17}$$

2.2. *Approximate solution*

For the flange of a metal sheet shown in Fig. 2, the following boundary conditions can be given when the flange is considered as an annular plate mentioned above:

$$w \geqslant 0, \quad \rho_{\#} \leqslant \rho \leqslant 1 \tag{18a}$$

$$w = 0, \quad \rho = \rho_{\#} \tag{18b}$$

$$\frac{\partial^2 w}{\partial\rho^2} + \nu_{\#}\Box w = 0, \quad \rho = \rho_{\#} \text{ and } 1 \tag{18c}$$

where

$$\nu_{\#} = (1-\phi E_s^0)/2 \tag{19}$$

and

$$\rho_{\#} = a/b. \tag{20}$$

Assuming that eqn (13) has a solution of the form

$$w(\rho,\theta) = f(\rho)g(\theta) \tag{21}$$

then according to the observation from experiment (Fig. 2) and condition (18a), we can take $g(\theta) = 1+\cos(n\theta)$, where n is the number of waves. Therefore, the following equation can be obtained easily by applying the well-known Kantorovich method:

$$D_{\#}L_1(f) - L_2(f) = 0 \tag{22}$$

where L_1 and L_2 are differential operators, i.e.

$$L_1 = 3\frac{d^4}{d\rho^4} + \frac{6}{\rho}\frac{d^3}{d\rho^3} - \frac{3+2n^2}{\rho^2}\frac{d^2}{d\rho^2} + \frac{3+2n^2}{\rho^3}\frac{d}{d\rho} + \frac{n^2(n^2-4)}{\rho^4} \tag{23}$$

$$L_2 = -\left\{3\ln(\rho)\frac{d^2}{d\rho^2} + \frac{3}{\rho}[1+\ln(\rho)]\frac{d}{d\rho} + \frac{n^2}{\rho^2}[\ln(\rho)-1]\right\}. \tag{24}$$

It is difficult to obtain the exact solution of eqn (22), although it is an ordinary differential equation. Hence, we use the Galerkin method to get its approximate solution. To do so, we take

$$f(\rho) = c(\rho^{\lambda_1} - \rho_{\#}^{\lambda_3}\rho^{\lambda_2}). \tag{25}$$

This expression satisfies conditions (18a) and (18b), and meets condition (18c) in the sense of the Kantorovich approach. In eqn (25), c is an undetermined coefficient and

$$\begin{cases} \lambda_1 = \frac{1}{2}(1-\nu_{\#}+\lambda_3) \\ \lambda_2 = \frac{1}{2}(1-\nu_{\#}-\lambda_3) \\ \lambda_3 = [(1-\nu_{\#})^2 + \frac{4}{3}\nu_{\#} n^2]^{1/2} \end{cases} \tag{26}$$

are the constants dependent on the properties of the plate material. Substitution of eqn (25) into the Galerkin equation, eqn (22), leads to

$$D_{\#}\int_{\rho_{\#}}^{1} L_1(f)(\rho^{\lambda_1} - \rho_{\#}^{\lambda_3}\rho^{\lambda_2})\, d\rho - \int_{\rho_{\#}}^{1} L_2(f)(\rho^{\lambda_1} - \rho_{\#}^{\lambda_3}\rho^{\lambda_2})\, d\rho = 0. \tag{27}$$

To obtain the non-trivial solution of eqn (22) as the form of eqn (25), it is necessary to take

$$D_{\#}\int_{\rho_{\#}}^{1} L_1(\rho^{\lambda_1} - \rho_{\#}^{\lambda_3}\rho^{\lambda_2})(\rho^{\lambda_1} - \rho_{\#}^{\lambda_3}\rho^{\lambda_2})\, d\rho - \int_{\rho_{\#}}^{1} L_2(\rho^{\lambda_1} - \rho_{\#}^{\lambda_3}\rho^{\lambda_2})(\rho^{\lambda_1} - \rho_{\#}^{\lambda_3}\rho^{\lambda_2})\, d\rho = 0. \tag{28}$$

Considering that

$$D_{\#} = \frac{E_s^0}{Y}\frac{t^2}{b^2}\,[3(1+\phi E_s^0)(3-\phi E_s^0)]$$

eqn (28) can be rewritten as

$$\zeta = \sqrt{\left(\frac{E_s^0}{Y}\right)}\frac{t}{b} = \left[3(1+\phi E_s^0)(3-\phi E_s^0)\frac{\int_{\rho_{\#}}^{1} L_2(\rho^{\lambda_1} - \rho_{\#}^{\lambda_3}\rho^{\lambda_2})(\rho^{\lambda_1} - \rho_{\#}^{\lambda_3}\rho^{\lambda_2})\, d\rho}{\int_{\rho_{\#}}^{1} L_1(\rho^{\lambda_1} - \rho_{\#}^{\lambda_3}\rho^{\lambda_2})(\rho^{\lambda_1} - \rho_{\#}^{\lambda_3}\rho^{\lambda_2})\, d\rho}\right]^{1/2} \tag{29}$$

Equation (29) is similar in form to eqns (25) and (31) of Ref. [4]. By applying eqns (28) and (29), for different geometrical and physical parameters of annular plates, E_s^0 can be solved directly and therefore the critical condition of wrinkling is obtained.

3. DISCUSSION AND CONCLUSIONS

To illustrate the application of the above results to practice, numerical examples are shown below. The ratio of elastic modulus to yield stress, E/Y, is taken to be 500 and Poisson's ratio to be 0.3. The value of ζ can be calculated from eqn (29), and E_s^0 can also

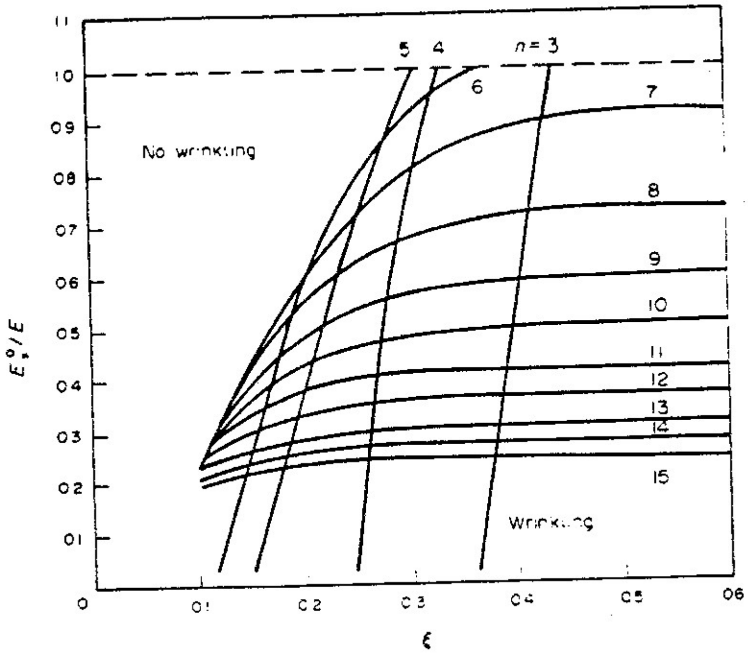

Fig. 3. The critical values of E_s^0/E.

be obtained for every given t/b. The well-known Powell method[6] is used as the solver of eqn (29). The results shown in Fig. 3 are calculated for $t/b = 0.02$.

The non-dimensional values of E_s^0/E corresponding to different wrinkling modes are shown in Fig. 3, and the curves of ζ are plotted on Fig. 4 against ξ. The regions above the dashed line in Fig. 3 and on the right-hand side of the chain line in Fig. 4 are those where plastic wrinkling cannot occur. Figure 3 indicates that the value of E_s^0/E almost remains constant in a relatively large interval of ξ. Figure 4 shows directly that for a smaller ξ, the corresponding wave number n must be large once wrinkling takes place. That is to say, the

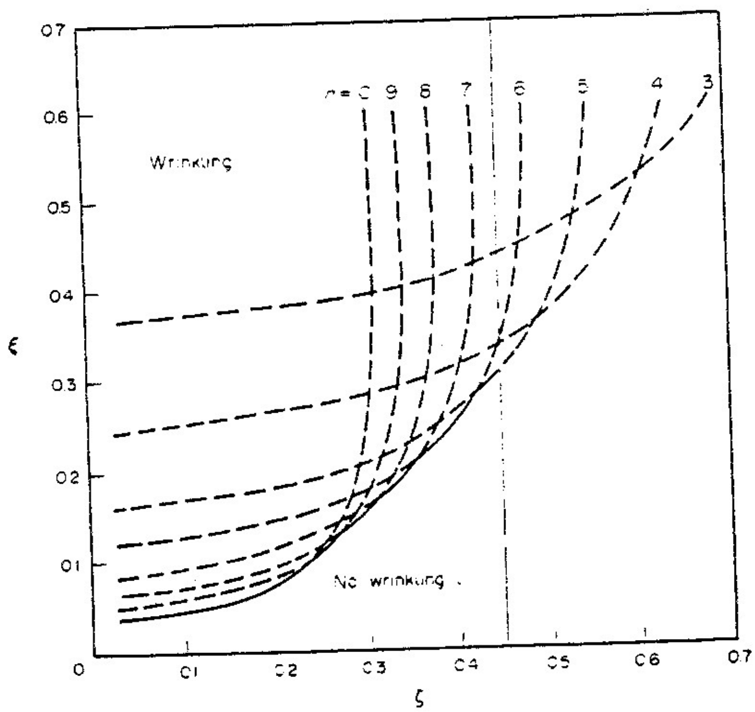

Fig. 4. The critical curves of annular plates.

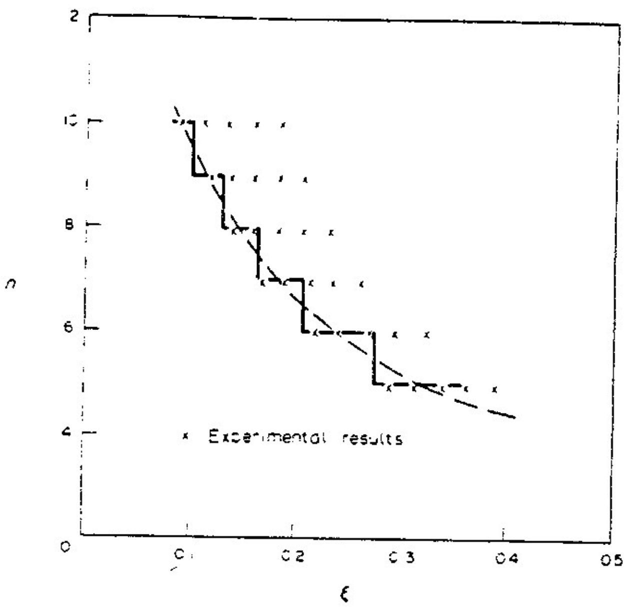

Fig. 5. The variation of wave number n with ξ.

wrinkling modes corresponding to a small wave number appear only if the width of those flanges is large. Figures 3 and 4 also show that the values of ζ of two neighbouring wrinkling modes approach each other as ξ decreases. This fact tells us that it is impossible to predict exactly the wave number when wrinkling occurs at a lower value of ξ because there exist many disturbances during deep-drawing operation. Figure 5 compares the present results with those from experiments[9]. It also supports the above conclusions. Figure 5 indicates that the theoretical results of this paper are in good agreement with experimental values. More accurate solutions can be obtained by the present method when better approximate functions $f(\rho)$ and $g(\theta)$ are taken.

In this paper, a static criterion is applied to predict wrinkling. It is a well-established fact that non-conservative systems should be analysed by the dynamical method. However, the validity of the present analysis is preliminarily confirmed by some experimental results. Of course, it is still open for further studies.

It follows from the procedure of the above analysis that the method provided in this paper is convenient and forthright and is very suitable for engineering applications. The approximate solution obtained is simple in form and is in good agreement with experiments.

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