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MODIFIED ADAPTIVE DYNAMIC RELAXATION METHOD AND ITS APPLICATION TO ELASTIC–PLASTIC BENDING AND WRINKLING OF CIRCULAR PLATES

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Abstract—Based on the adaptive Dynamic Relaxation (aDR) method, a modified adaptive Dynamic Relaxation (maDR) method is proposed which is more efficient than the former in solving non-linear problems. It is then applied to analysing the elastic–plastic bending of circular plates in large deflection and their following wrinkling, and leads to satisfactory results compared with corresponding experimental ones. It shows that the maDR method possesses vast vistas of applications to engineering problems.

NOTATION

b	radius of a circular plate
C	damping matrix
c	damping factor
d	deflection of the circular plate at the position on mid-plane where a ring load exerts
E	modulus of elasticity
e_{ij}	deviatoric strain
$\mathbf{F}$	vector of generalized external force
G	shear modulus
H	material hardening modulus
$\mathbf{K}$	stiffness matrix
M	bending moment
$\mathbf{M}$	mass matrix
m_{ij}	the element of $\mathbf{M}$
N	membrane force
n	number of iteration with respect to fictitious time
P	total external load
$\mathbf{P}$	vector of internal force
p	density of the ring load
q	density of the uniform pressure for circular plates, or of external load in general
r	radial coordinate
r_p	radius of the ring load
s_{ij}	deviatoric stress
u, v, w	radial, circumferential and vertical (in z direction) dispalcements, respectively, on the mid-plane of the plate
w_0	central deflection of the plate
$\mathbf{X}$	vector of generalized solution
$\dot{\mathbf{X}}, \ddot{\mathbf{X}}$	vectors of fictitious velocity and acceleration, respectively
Y	material yield stress
z	vertical coordinate
γ	shear strain
$\delta(\cdots)$	a small increment of $(\cdots)$
ϵ	radial or circumferential strain
θ	circumferential coordinate
κ	curvature of the plate
ν	Poisson's ratio
σ	stress
τ	increment of fictitious time
ϕ	Mises yield surface

1. INTRODUCTION

In order to analyse various complicated problems in engineering, many kinds of efficient numerical methods such as finite difference method, finite element method and the weighted residual method have been developed. However, the accompanying problem is that large computers are needed to solve the related large scale equations. Sometimes, the equations are so large that one can only obtain rough results. This is especially conspicuous in solving non-linear problems. In addition, numerical instability during iteration is often involved.

In the traditional methods of solving equations from static equilibrium problems, it is considered that internal forces exist initially in the structures. In so doing, one assumes that the external forces were exerted very slowly so that the dynamic process of the structures could be neglected. In fact, as has been pointed out by Rayleigh [1], the static solution of a mechanics system can be referred to as the steady state part of the transient response of the system to step loading. This approach was successfully applied to solving linear problems by Otter [2] and Day [3] independently in 1965, and was named the Dynamic Relaxation (DR) method.

Nowadays, researchers are attracted by the efficiency of solving non-linear problems with DR. The applications of DR to various problems indicate that the method has the following distinctive features (see, for example, [4–7]).

(a) The scheme of its algorithm is fixed so that the programing becomes straightforward.

(b) There is no need to solve large scale equations directly but, instead, to obtain solutions with simple explicit iterations, which makes it possible to solve complicated problems using a microcomputer.

(c) It is incredibly reliable and indefatigable in seeking an equilibrium state.

Underwood [8] summarized the advances of the DR method up to 1982 and proposed an adaptive Dynamic Relaxation method. It is a typical paper on DR.

The present paper, based on the aDR method, proposes a modified adaptive Dynamic Relaxation (maDR) method which is more efficient than the former. Compared with aDR, the latter possesses advantages of higher rate of convergency and less storage of intermediate data. Moreover, by making use of the character of maDR, the paper overcomes naturally the main difficulties in applications of the well-known dynamic criterion on stability, and successfully predicts the plastic wrinkling loads of circular plates during axisymmetric bending. The applications of the present method to the analysis of elastic–plastic bending and wrinkling of circular plates, which couple the non-linearities from geometry and material behaviour, indicate that the maDR method has broad prospects of applications to engineering problems.

2. THE maDR METHOD

The governing equations describing a static mechanics problem can be expressed as

$$\mathbf{P}(\mathbf{X}^*) = \mathbf{F}, \tag{1}$$

where $\mathbf{X}^*$ is the vector of actual generalized solution of the problem. If instead of $\mathbf{X}^*$, an approximate solution, $\mathbf{X}$, is substituted into (1), a residual due to disequilibrium, $\mathbf{R} = \mathbf{F} - \mathbf{P}(\mathbf{X})$, appears. It is considered that the residual force vector leads to the movement of the system so that the corresponding dynamic equations become

$$\mathbf{M}\ddot{\mathbf{X}} + \mathbf{C}\dot{X} + \mathbf{P} = \mathbf{F}, \tag{2}$$

where $\mathbf{M}$ and $\mathbf{C}$ are fictitious mass and damping matrices. The word 'fictitious' indicates that the dynamic process described by (2) is fictitious as the maDR method is used so that $\mathbf{M}$ and $\mathbf{C}$ can be artificially chosen to obtain the static solution in a minimum number of pseudo-time increment steps. Therefore, one often chooses them as diagonal ones and makes $\mathbf{C} = c\mathbf{M}$. In so doing, one obtains explicit iteration formulas for solving the approximate solution vector, $\mathbf{X}$, when the central difference scheme with respect to pseudo-time is used, i.e.

$$\dot{\mathbf{X}}^{n+1/2} = \frac{2-\tau^n c^n}{2+\tau^n c^n}\dot{\mathbf{X}}^{n-1/2} + \frac{2\tau^n}{2+\tau^n c^n}\mathbf{M}^{-1}\mathbf{R}^n$$

$$\mathbf{X}^{n+1} = \mathbf{X}^n + \tau^{n+1}\dot{\mathbf{X}}^{n+1/2}, \tag{3}$$

where τ^n is the pseudo-time increment of the nth iteration,

$$\mathbf{R}^n = \mathbf{F} - \mathbf{P}(\mathbf{X}^n)$$

$$\dot{\mathbf{X}}^{n-1/2} = (\mathbf{X}^n - \mathbf{X}^{n-1})/\tau^n$$

and

$$\ddot{\mathbf{X}}^n = (\dot{\mathbf{X}}^{n+1/2} - \dot{\mathbf{X}}^{(n-1/2)})/\tau^n.$$

Obviously, as $\mathbf{M}$ is diagonal, (3) is algebraic. That is, each solution vector component may be computed individually.

The differences between the aDR and maDR methods are the calculations of c^n and $\mathbf{X}^0$ in iterations (3). In the latter, c^n is calculated by

$$c^n = 2\left\{\frac{(\mathbf{X}^n)^t\mathbf{P}(\mathbf{X}^n)}{(\mathbf{X}^n)^t\mathbf{M}\mathbf{X}^n}\right\}^{1/2}$$

and each element of $\mathbf{X}^0$, x_i^0, is obtained in such a way that $x_i^0 = (x_i^* + x_i^{**})/2$, where x_i^* and x_i^{**} are the values of two neighbouring but opposite peaks of locus of x_i detected in the manner of $c = 0$. Furthermore, to guarantee the numerical stability, the element of $\mathbf{M}$ is determined by the Gerschgörin theorem as

$$m_{ii} \geq \tfrac{1}{4}(\tau^n)^2 \sum_j |k_{ij}|, \tag{4}$$

where k_{ij} is the element of $\mathbf{K}$ which is calculated by

$$\mathbf{K} = \frac{\partial \mathbf{P}(\mathbf{X})}{\partial \mathbf{X}}.$$

Then, the algorithm of the maDR method can be expressed as:

(a) compute $\mathbf{M}$; $\mathbf{X}^0 = \dot{\mathbf{X}}^0 = \mathbf{0}$; $c^0 = 0$,
(b) determine $\bar{\mathbf{X}} = (\mathbf{X}^* + \mathbf{X}^{**})/2$,
(c) $\mathbf{X}^0 = \bar{\mathbf{X}}$ and $\dot{\mathbf{X}}^0 = \mathbf{0}$,
(d) e_{KE}, e_{RR} and N given; $n = 0$,
(e) calculate $\mathbf{M}$ again,
(f) compute $\mathbf{R}^n$,
(g) if $|R_i^n| \leq e_{RR}$ stop, otherwise continue,
(h) calculate $\mathbf{X}^{n+1/2}$ and $c^n(\dot{\mathbf{X}}^{1/2} = \tau^0\mathbf{M}^{-1}\mathbf{R}^0/2)$,
(i) if $\Sigma_j(\dot{x}_j^{n+1/2})^2 \leq e_{KE}$ stop, otherwise continue,
(j) determine $\mathbf{X}^{n+1}$,
(k) exert boundary conditions,
(l) $n = n + 1$,
(m) if $n \geq N$ stop, otherwise return to (e).

No numerical instability was found during iterations of the above algorithm. Moreover, the comparisons of two examples in the present paper show that the present method may save 10% of iteration steps compared with aDR. Throughout the present calculations, $e_{KE} = 1.E - 10 \sim 1.E - 15$ and $e_{RR} = 1.E - 6$ were taken. The values of N in different examples were different.

Difficulties usually exist in the applications of the dynamic criterion. The main problems are that:

(i) an initial value problem will be treated as one uses the criterion, while it is difficult for the usual methods of solving static problems to attach such a function; and the criterion relates to any initial disturbances as well as to the bounded property of the mechanics system as time $t \to \infty$, and

(ii) various movements due to disturbances will produce complications of loading and unloading.

Fortunately, the maDR method can naturally combine with the dynamic criterion. Firstly, the maDR method itself solves static problems by transforming them into corresponding dynamic ones. Hence, there does not exist any difficulty in producing dynamic disturbances. Moreover, these disturbances possessed by maDR are free, and one can also observe the bounded property of a system over a long time. Secondly, one should first overcome the difficulties from loading and unloading during iteration as maDR is chosen to solve an elastic–plastic problem, so that problem (ii) will also be resolved.

It is easy to write an algorithm on detecting plastic wrinkling of circular plates during their axisymmetric bending according to the above statement. Obviously, one should only observe the bounded property of the incremental circumferential displacement, δv, because before wrinkling $\delta v = 0$.

3. BENDING AND WRINKLING OF CIRCULAR PLATES

The elastic–plastic bending in large deflection and the following wrinkling of circular plates subjected

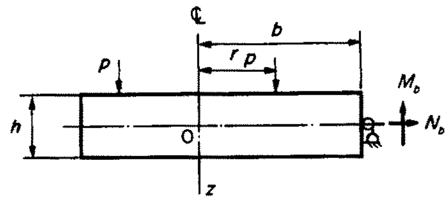

Fig. 1. Illustration of loading and boundary conditions.

to ring load and of those subjected to uniform distributed pressure are to be analysed. The boundary conditions of the latter are simply supported or clamped ones, but those of the former are shown in Fig. 1 where

$$N_b = \frac{pr_p(\cos\theta - \mu \sin\theta)}{(b + u_b)(\sin\theta + \mu\cos\theta)}$$

and

$$M_b = N_b h/2.$$

In the above expressions, u_b is the radial displacement on $z = h/2$ at the plate periphery, and $\mu = 0.3$ and $\theta = 20°$ are taken throughout the calculations. In fact, the plate shown in Fig. 1 is a mechanics model of the specimen in a conical cup test [9] which is one of the standardized tests in research on sheet metal forming.

Under the cylindrical coordinate system, the equilibrium equations of incremental form can be written as

$$\frac{\partial \delta N_r}{\partial r} + \frac{1}{r}\frac{\partial \delta N_{r\theta}}{\partial \theta} + \frac{1}{r}(\delta N_r - \delta N_\theta) = 0$$

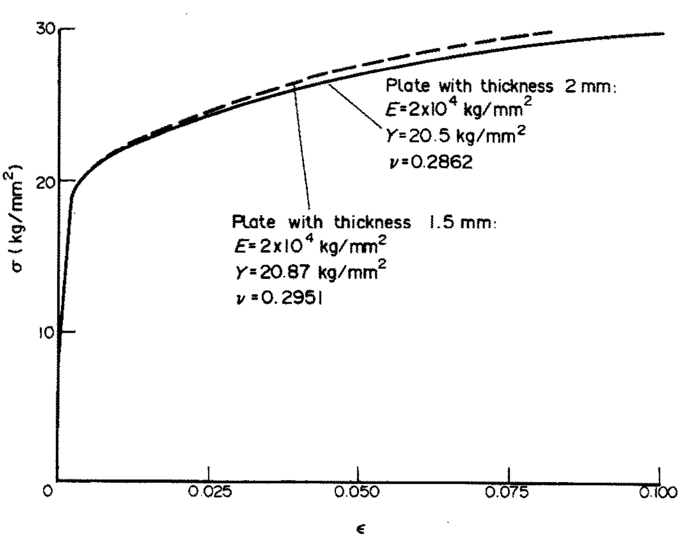

Fig. 2. Stress–strain curves of plate materials.

$$\frac{\partial \delta N_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial \delta N_\theta}{\partial \theta}+\frac{2}{r}\delta N_{r\theta}=0$$

$$\frac{\partial^2 \delta M_r}{\partial r}+\frac{2}{r}\frac{\partial \delta M_r}{\partial r}+\frac{1}{r}\frac{\partial \delta M_\theta}{\partial r}\frac{2}{r}\frac{\partial^2 \delta M_{r\theta}}{\partial r\,\partial\theta}$$

$$+\frac{2}{r^2}\frac{\partial \delta M_{r\theta}}{\partial \theta}+\frac{1}{r^2}\frac{\partial^2 \delta M_\theta}{\partial \theta^2}+\left\{\frac{\partial^2 \delta w}{\partial r^2}(N_r+\delta N_\theta)\right.$$

$$+\frac{\partial^2 w}{\partial r^2}\delta N_r+\frac{2}{r}\left[\frac{\partial^2 \delta w}{\partial r\,\partial\theta}(N_{r\theta}+\delta N_{r\theta})+\frac{\partial^2 w}{\partial r\,\partial\theta}\delta N_{r\theta}\right]$$

$$-\frac{2}{r^2}\left[\frac{\partial \delta w}{\partial \theta}(N_{r\theta}+\delta N_{r\theta})+\frac{\partial w}{\partial\theta}\delta N_{r\theta}\right]$$

$$+\left(\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)\delta N_\theta$$

$$\left.+\left(\frac{1}{r}\frac{\partial \delta w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 \delta w}{\partial \theta^2}\right)(N_\theta+\delta N_\theta)\right\}+\delta q=0.$$

The relations between strains and displacements are

$$\delta\epsilon_r^0=\frac{\partial \delta u}{\partial r}+\frac{\partial w}{\partial r}\frac{\partial \delta w}{\partial r}+\frac{1}{2}\left(\frac{\partial \delta w}{\partial r}\right)^2$$

$$\delta\epsilon_\theta^0=\frac{\partial u}{r}+\frac{1}{r}\frac{\partial \delta v}{\partial\theta}\frac{1}{r^2}\frac{\partial w}{\partial \theta}+\frac{\partial \delta w}{\partial\theta}+\frac{1}{2}\left(\frac{1}{r}\frac{\partial \delta w}{\partial \theta}\right)^2$$

$$\delta\gamma_{r\theta}^0=\frac{1}{r}\frac{\partial \delta u}{\partial \theta}+\frac{\partial \delta v}{\partial r}-\frac{\partial v}{r}\frac{1}{r}\frac{\partial \delta w}{\partial r}\left(\frac{\partial w}{\partial\theta}+\frac{\partial \delta w}{\partial \theta}\right)$$

$$+\frac{\partial w}{\partial r}\left(\frac{1}{r}\frac{\partial \delta w}{\partial \theta}\right)$$

$$\delta\kappa_r=-\frac{\partial^2 \delta w}{\partial r^2}$$

$$\delta\kappa_\theta=-\left(\frac{1}{r}\frac{\partial \delta w}{\partial r}+\frac{1}{r^2}\frac{\partial^2\delta w}{\partial\theta^2}\right)$$

$$\delta\kappa_{r\theta}=-\frac{1}{r}\frac{\partial^2 \delta w}{\partial r\,\partial\theta}+\frac{1}{r^2}\frac{\partial \delta w}{\partial \theta}$$

$$\delta\epsilon_r=\delta\epsilon_r^0+z\delta\kappa_r$$

$$\delta\epsilon_\theta=\delta\epsilon_\theta^0+z\delta\kappa_\theta$$

$$\delta\epsilon_{r\theta}=\delta\epsilon_{r\theta}^0+2z\delta\kappa_{r\theta}.$$

The simple J_2 theory of plasticity is used; that is, in the elastic region:

$$\delta\epsilon_{ij}=\frac{\partial\sigma_{ij}}{2G}-\frac{v}{E}\delta\sigma_{kk}\delta_{ij};$$

in the plastic region:

$$\delta e_{ij}=\frac{\partial s_{ij}}{2G}+\delta\lambda\frac{\delta\phi}{\sigma_{ij}}$$

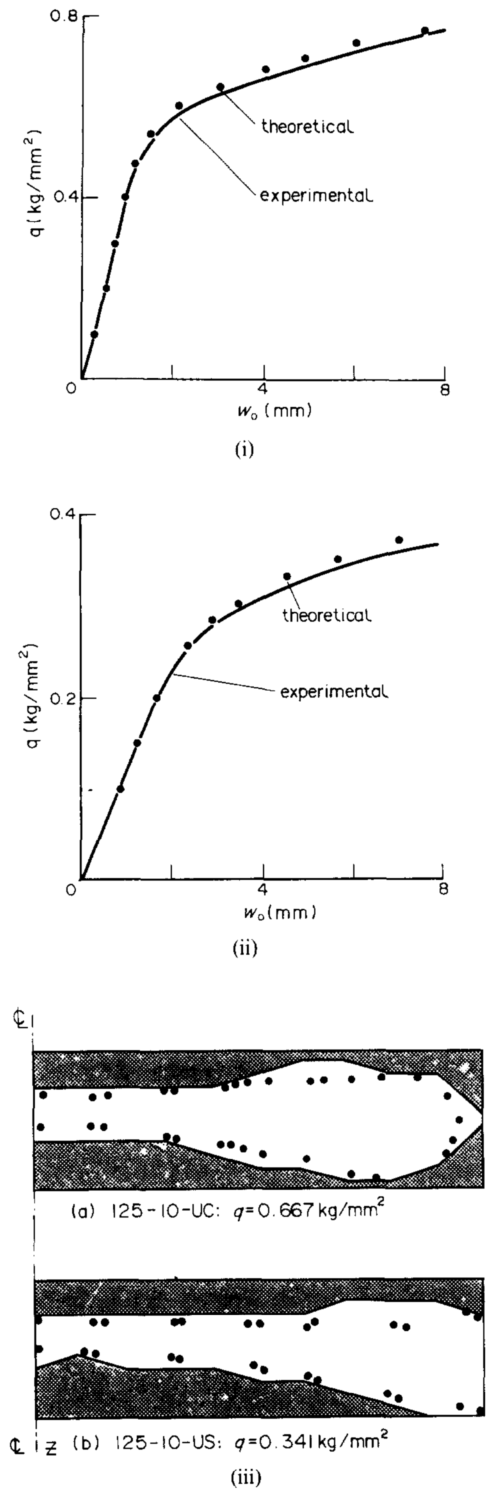

Fig. 3. (i) Load–central deflection curve of 125–10–UC; (ii) load–central curve of 125–10–US; (iii) distributions of plastic regions (●: experimental; ——: theoretical).

$$\delta\epsilon_{kk}=\frac{1-2v}{E}\delta\sigma_{kk}$$

$$\delta\lambda=\begin{cases}0 & \phi(\sigma_{ij}+\delta\sigma_{ij})\leqslant 0\\ H\delta\phi & \phi(\sigma_{ij}+\delta\sigma_{ij})>0.\end{cases}$$

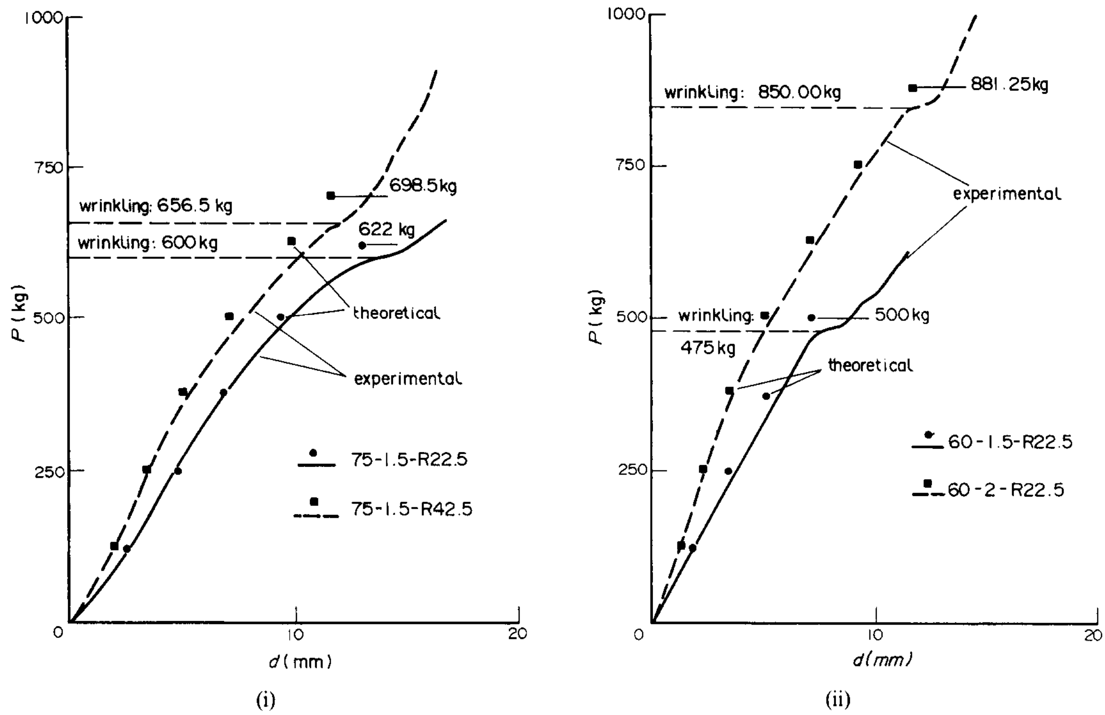

Fig. 4. Load–deflection curves. (i) 75–1.5–$R^{22.5}_{42.5}$; (ii) 60–$^{1.5}_{2.0}$–R22.5.

When the above governing equations are changed into corresponding finite difference ones, the approximate solution can be obtained easily by the algorithm described in the previous section. In fact, before wrinkling, the deformation of a plate is axisymmetric so that the equations may be simplified significantly.

4. NUMERICAL RESULTS AND DISCUSSION

The numerical results from the maDR method are compared systematically to the experimental ones. For the convenience of discussion, the following notation is adopted: (i)–(ii)–(iii)(iv); representing (i) the ratios of the plate (mm); (ii) the thickness of the plate; (iii) the type of loading, ring load (R) or uniform pressure (U); (iv) number (mm), the radius of ring load, S(C), simply supported (clamped) plate to uniform transverse pressure. For example, 75–1.5–R22.5 stands for the plate of radius 75 mm, thickness 1.5 mm and subjected to a ring load with radius 22.5 mm.

For the plates loaded by ring loads, the corresponding experiments were carried out by the authors so that the real stress–strain curves of the plate materials could be used in the present calculations (see Fig. 2), while for those subjected to uniform pressure, experimental results were taken from [10], therefore in our analysis, the coefficients of material were also identified with the paper: $E = 2.04 \times 10^4$ kg/mm^2, $\nu = 0.28$, $Y = 34.5$ kg/mm^2 and the material was considered as an elastic–perfectly plastic one.

Figures 3 and 4 show that the numerical results from the maDR method are in very good agreement with experimental ones in all the cases of determining axisymmetric deformation states, detecting wrinkling loads and revealing the developments of plastic regions. The accuracy of the numerical results can also be shown quantitatively, for example, by the errors of wrinkling loads, which are 5.26, 3.68, 3.67 and 6.4% for the cases 60–1.5–R22.5, 60–2.0–R22.5, 75–1.5–R22.5 and 75–1.5–R42.5, respectively.

5. CONCLUDING REMARKS

The efficiency and potential of the maDR method applied to elasto-plastic deformation and bifurcation have been shown by the above results. However, it should be pointed out that other approaches of improvement of the DR method should also be brought to general attention, such as the semi-explicit approach and the combination with other iteration methods.

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