Int. J. Mech. Sci. Vol. 28, No. 11, pp. 729–737, 1986
Printed in Great Britain.

0020–7403/86 $3.00 + .00
Pergamon Journals Ltd.

THE ELASTIC WRINKLING OF AN ANNULAR PLATE UNDER UNIFORM TENSION ON ITS INNER EDGE

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(*Received* 7 *July* 1986)

Abstract—This paper analyses the elastic wrinkling of an annular plate subjected to in-plane uniform tensile stress on its inner edge with the combined use of the Kantorovich method and Galerkin method, and discusses the appearance of wrinkles on the flange of a metal circular sheet during its axisymmetric deep-drawing operation.

NOTATION

a	inner radius of an annular plate
b	outer radius of the annular plate
c_i	undetermined coefficients in approximate wrinkling mode
D	bending rigidity of plate
E	Young's modulus
f_0	a function of non-dimensional radial co-ordinate ρ
g_0, g_0^*, g_1	functions of circumferential co-ordinate θ
h	thickness of plate
L_1, L_2	differential operators
n	number of waves
r, θ	polar co-ordinates
w_0	mode of wrinkling
δ	the parameter defined by equation (12)
v	Poisson's ratio
ξ	non-dimensional parameter, $1 - a/b$
ξ_1	the parameter defined by equation (3)
ρ	non-dimensional radial co-ordinate, r/b
σ, τ	stresses
Φ_0	the function of ρ, defined by equation (13b)
ψ, ω	the parameters defined by equation (11)

1. INTRODUCTION

When the inner edge of an annular plate is loaded by an in-plane uniform tensile stress, the stress state of the plate, according to the theory of elasticity, can be expressed as

$$\sigma_r = \frac{\sigma_a a^2}{b^2 - a^2}\left(\frac{b^2}{r^2} - 1\right),$$
$$\sigma_\theta = -\frac{\sigma_a a^2}{b^2 - a^2}\left(\frac{b^2}{r^2} + 1\right), \quad (1)$$
$$\tau_{r\theta} = 0.$$

As the circumferential stress, σ_θ, is compressive and reaches its maximum amplitude on the inner edge of the plate, see Fig. 1, elastic buckling will first occur circumferentially, when the uniform boundary stress σ_a reaches a critical value. Here it is assumed that the plate has not yielded. This appearance is called *elastic wrinkling* (Fig. 2).

Investigation of wrinkling has a great deal of significance in many branches of mechanical engineering, especially in the axisymmetric deep-drawing process of circular plates. Engineers require that the flange of a workpiece in its deep-drawing operation should yield before elastic wrinkling occurs because otherwise it will impair the quality of the product. For this reason the wrinkling of an annular plate has been the focus of much research attention over many decades. Geckeler [1] simplified the problem and treated it by a one-dimensional

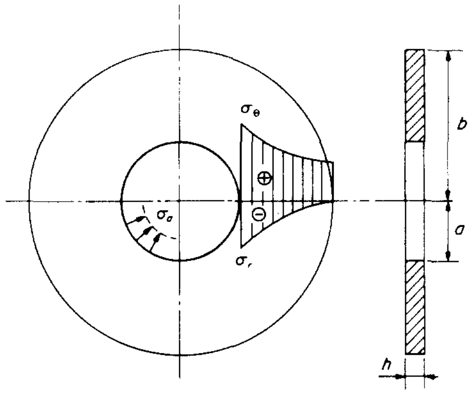

FIG. 1. An annular plate subjected to a uniform tensile stress on its inner edge.

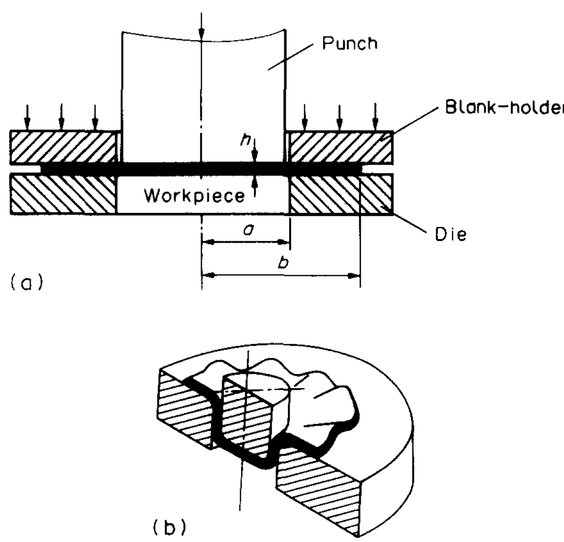

FIG. 2. (a) A typical deep-drawing device. (b) The wrinkling of a flange.

model, furnishing some formulae to predict the critical circumferential stress and the number of waves. His model was employed and extended by many other experts afterward [2–4]. However, Yu and Johnson [5] realized the limitation of Geckeler's one-dimensional model and studied the problem using a two-dimensional one by means of the energy method. However, their results may still be too simplistic for general application.

The principal weakness of the approximate analyses as outlined above is that they do not consider the effect of elastic wrinkling of the workpiece apart from that by Yu and Johnson in Ref. [5]. Furthermore, available solutions about the elastic buckling of annular plates [6–8] cannot be applied to the deep-drawing operation. In order to make up for this weakness we now furnish some relatively accurate analytical results for the deep-drawing operation and discuss the method to solve wrinkling. The present paper analyzes the title problem with the combined use of the methods of Kantorovich and Galerkin [9] and suggests approaches to improve the accuracy of the approximate solutions. In addition, the way to overcome the appearance of elastic wrinkling in the deep-drawing process is discussed.

2. SOLUTION

2.1 *Critical circumferential stresses*

Consider an annular plate with inner radius a, outer radius b and thickness h. The inner edge of the plate is subjected to a uniform in-plane tensile stress σ_a, see Fig. 1. The differential equation for the buckling of such a plate in polar co-ordinates is [10]

$$D\nabla_r^4 w - h\left[\sigma_r \frac{\partial^2 w}{\partial r^2} + 2\tau_{r\theta}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial \theta}\right) + \sigma_\theta\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)\right] = 0, \tag{2}$$

where $D = Eh^3/12(1-\nu^2)$ is the bending rigidity, w is the deflection describing the wrinkling for the middle plane and the operator ∇_r^4 is defined by

$$\nabla_r^4 = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)^2.$$

Substituting expression (1) into equation (2) and letting

$$\xi_1 = -\frac{a^2 b^2 h \sigma_a}{D(b^2 - a^2)}, \tag{3}$$

we get

$$\nabla_r^4 w + \xi_1\left(\frac{1}{r^2}\nabla_{lr}^2 w - \frac{1}{b^2}\nabla_r^2 w\right) = 0, \tag{4}$$

where

$$\nabla_{lr}^2 = \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

By using a non-dimensional co-ordinate $\rho = r/b$, equation (4) can be re-written as

$$\nabla^4 w + \xi_1\left(\frac{1}{\rho^2}\nabla_1^2 w - \nabla^2 w\right) = 0, \tag{5}$$

in which

$$\nabla^4 = \nabla^2 \cdot \nabla^2,$$

$$\nabla^2 = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \theta^2}$$

and

$$\nabla_1^2 = \frac{\partial^2}{\partial \rho^2} - \frac{1}{\rho}\frac{\partial}{\partial \rho} - \frac{1}{\rho^2}\frac{\partial^2}{\partial \theta^2}.$$

To obtain an approximate solution of equation (5), let

$$w = w_0(\rho, \theta) = f_0(\rho) g_0(\theta), \tag{6}$$

where $g_0(\theta)$ should be a periodic function according to experimental observations. First, we consider an annular plate with a simply supported inner edge and a free outer edge and take

$$g_0(\theta) = \cos(n\theta), \quad (n = 0, 1, 2, \ldots), \tag{7}$$

where n is the number of waves in a given wrinkling mode. Obviously, the case $n = 0$ corresponds to the axisymmetric buckling mode. With the help of the well-known Kantorovich method [9] we obtain the following ordinary differential equation from equation (5)

$$f_0^{(4)} + \frac{2}{\rho} f_0''' + \left(\frac{\xi_1 - 2n^2 - 1}{\rho^2} - \xi_1\right) f_0'' - \frac{1}{\rho}\left(\frac{\xi_1 - 2n^2 - 1}{\rho^2} - \xi_1\right) f_0'$$
$$+ \frac{n^2}{\rho^2}\left(\frac{\xi_1 + n^2 - 4}{\rho^2} + \xi_1\right) f_0 = 0. \tag{8}$$

If $n = 0$ or $b \to \infty$, the above equation can be simplified and exact solutions are then obtained (see the Appendix). To derive an approximate solution of equation (8), we use the well-known Galerkin method.

The boundary conditions of this problem can be expressed as,

at $\rho = a/b$, $$f_0 = 0, \tag{9a}$$

at $\rho = 1$ and a/b, $$f_0'' + \frac{\nu}{\rho} f_0' - \frac{\nu n^2}{\rho^2} f_0 = 0, \tag{9b}$$

at $\rho = 1$, $$f_0''' + \frac{1}{\rho} f_0'' - [(1+2n^2 - \nu n^2)/\rho^2] f_0' + [n^2(3-\nu)/\rho_1^3] f_0 = 0. \tag{9c}$$

From the loading state of the plate the radial resultant bending moment M_r should be zero through the plate. Therefore we may take

$$f_0(\rho) = \rho^{\psi}[c_1 \cos(\omega \ln \rho) + c_2 \sin(\omega \ln \rho)], \tag{10}$$

in which

$$\psi = \tfrac{1}{2}(1-\nu) \quad \text{and} \quad \omega = \tfrac{1}{2}\sqrt{4\nu n^2 - (1-\nu)^2}, \tag{11}$$

where ν is Poisson's ratio for the material and ψ and ω are positive constants for metals (so long as $\nu \geqslant 0.17$). Expression (10) satisfies condition (9b) automatically. Substituting (10) into (9a) gives

$$c_2 = \delta c_1,$$

where

$$\delta = -\cot[\omega \ln(a/b)]. \tag{12}$$

Then, (10) can be rewritten as

$$f_0(\rho) = c_1 \rho^{\psi}[\cos(\omega \ln \rho) + \delta \sin(\omega \ln \rho)]. \tag{13a}$$

For the sake of convenience we release the condition (9c) for the time being [11]. Letting

$$f_0(\rho) \equiv c_1 \Phi_0(\rho) \equiv c_1 P(\rho) Q(\rho), \tag{13b}$$

where

$$P(\rho) = \rho^{\psi},$$
$$Q(\rho) = \cos(\omega \ln \rho) + \delta \sin(\omega \ln \rho), \tag{13c}$$

and

$$\Phi_0(\rho) = P(\rho) Q(\rho),$$

we get the following equation with the help of the Galerkin method

$$\int_{a/b}^{1} [L_1 f_0(\rho) + \xi_1 (L_2 f_0(\rho))] \Phi_0(\rho) d\rho = 0, \tag{14a}$$

i.e.

$$\int_{a/b}^{1} [L_1 \Phi_0 + \xi_1 (L_2 \Phi_0)] \Phi_0 d\rho c_1 = 0. \tag{14b}$$

The integral in the above equation must be zero if we require a non-trivial solution of form (13a). This yields

$$\xi_1 = -\frac{\int_{a/b}^{1} (L_1 \Phi_0) \Phi_0 d\rho}{\int_{a/b}^{1} (L_2 \Phi_0) \Phi_0 d\rho} \tag{15a}$$

or

$$\frac{\sigma_a b^2 h}{D} = \frac{b^2 - a^2}{a^2} \frac{\int_{a/b}^{1} (L_1 \Phi_0) \Phi_0 d\rho}{\int_{a/b}^{1} (L_2 \Phi_0) \Phi_0 d\rho}. \tag{15b}$$

In the above equations, L_1 and L_2 denote operators which can be expressed as

$$L_1 = \frac{d^4}{d\rho^4} + \frac{2}{\rho}\frac{d^3}{d\rho^3} - \frac{2n^2+1}{\rho^2}\left(\frac{d^2}{d\rho^2} - \frac{1}{\rho}\frac{d}{d\rho}\right) + \frac{n^2(n^2-4)}{\rho^4},$$

and

$$L_2 = \left(\frac{1}{\rho^2} - 1\right)\frac{d^2}{d\rho^2} - \frac{1}{\rho}\left(\frac{1}{\rho^2} + 1\right)\frac{d}{d\rho} + \frac{n^2}{\rho^2}\left(\frac{1}{\rho^2} + 1\right). \tag{16}$$

The approximate value of the circumferential stress, which is dependent on the dimensions of the annular plate, the number of waves and the elastic constants of the material, can easily be obtained by the substitution of equation (13) into (15).

In the case of the deep-drawing process, condition (9c) is no longer valid. However, the wrinkling mode must satisfy

$$w \geqslant 0. \tag{17}$$

For this reason we take

$$w_0 = f_0 g_0^* = f_0(\rho)(1 + g_0(\theta)), \tag{18}$$

where

$$1 + g_0(\theta) = 1 + \cos(n\theta) \geqslant 0$$

and

$$f_0(\rho) \geqslant 0.$$

The proof of the latter inequality will be found in the Appendix. Equation (15) still holds when a derivation similar to that mentioned above has been carried out, so long as L_1 and L_2 in the equation are replaced by the following expressions

$$L_1^* = 2\frac{d^4}{d\rho^4} + \frac{6}{\rho}\frac{d^3}{d\rho^3} - \frac{2n^2+3}{\rho^2}\left(\frac{d^2}{d\rho^2} - \frac{1}{\rho}\frac{d}{d\rho}\right) + \frac{n^2(n^2-4)}{\rho^4}$$

and

$$L_2^* = 2\left(\frac{1}{\rho^2} - 1\right)\frac{d^2}{d\rho^2} - \frac{2}{\rho}\left(\frac{1}{\rho^2} + 1\right)\frac{d}{d\rho} + \frac{n^2}{\rho^2}\left(\frac{1}{\rho^2} + 1\right). \tag{19}$$

2.2 *Yield condition*

For elastic–perfectly plastic material of yield stress Y, the Tresca yield condition gives

$$\sigma_r - \sigma_\theta = Y,$$

as the relation $\sigma_r > \sigma_z = 0 > \sigma_\theta$ is noted. Thus for yielding,

$$\frac{2\sigma_a a^2}{(b^2 - a^2)\rho^2} = Y,$$

and the left-hand side of the above equation reaches its maximum value when ρ attains a/b so that yield occurs first at the inner edge of the annular plate. This is in accordance with how wrinkling actually occurs. Thus the yield condition gives,

$$\frac{\sigma_a b^2 h}{D} = \frac{Yh(b^2 - a^2)}{2D}. \tag{20}$$

3. DISCUSSION AND CONCLUSIONS

The diagrams of non-dimensional critical stress, $\sigma_a b^2 h/D$, and the number of waves, n, against $\xi = 1 - a/b$ can be drawn easily from expression (15). In all calculations $\nu = 0.3$ is taken. Figure 3(a) and 4(a) are obtained using the wrinkling mode defined by equation (6), whilst Fig. 3(b) and 4(b) are derived from the mode given by equation (18). Figure 3 shows that the critical stress decreases gradually with increasing inner radius a when outer radius b is held constant. However, the number of waves, n, increases gradually in this case. It is obvious from the tendency of these critical curves that the values of the critical stresses corresponding

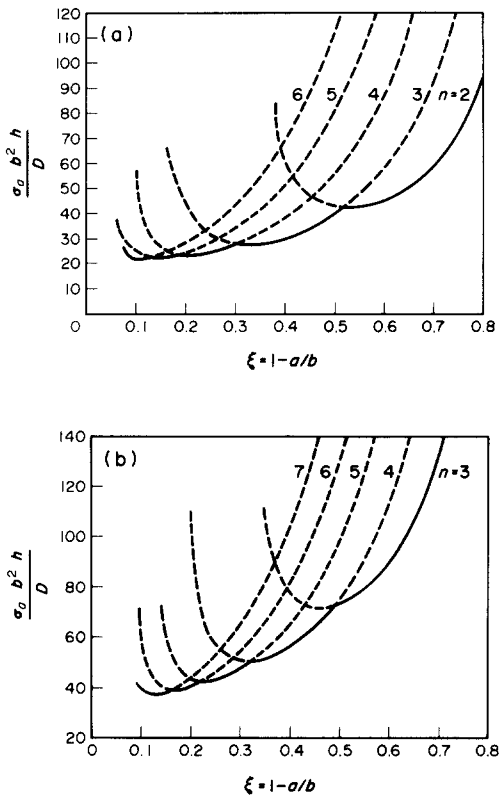

FIG. 3. Critical curves of annular plates: (a) results from wrinkling mode, equation (6); (b) results from wrinkling mode, equation (18).

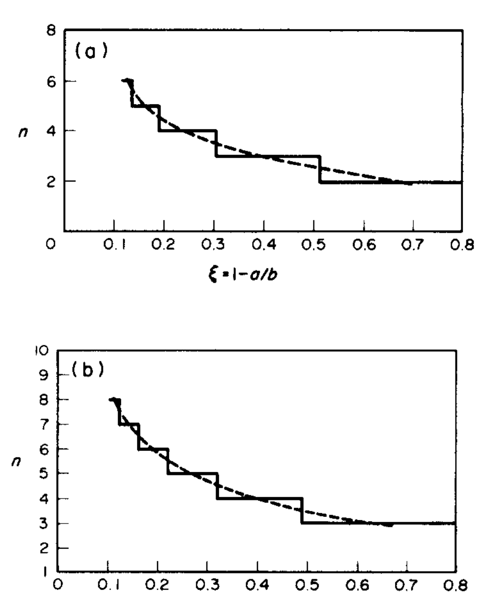

FIG. 4. The number of waves: (a) results from wrinkling mode, equation (6); (b) results from wrinkling mode, equation (18).

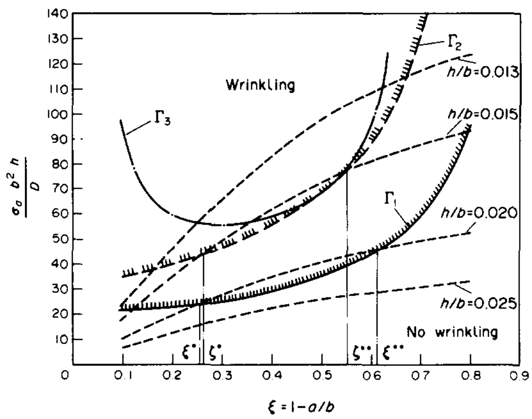

FIG. 5. A map showing wrinkling region and yielding region.

to the two neighbouring wave numbers are almost the same with increasing ratio a/b, see curves Γ_1 and Γ_2 in Fig. 5. It follows that for a sufficiently large value of a/b of the flange in the deep-drawing operation, the number n cannot be predicted accurately since there exist many disturbances during operation.

Generally, engineers hope that a flange will yield without the difficulties arising from elastic wrinkling. Figure 5 shows the restrictive conditions for guarantee of this requirement. Curves Γ_1 and Γ_2 are the critical curves of elastic wrinkling corresponding to the mode of equations (6) and (18), respectively. The region above the curve Γ_1 or Γ_2 for a different wrinkling mode is the wrinkling region and that below the curve corresponds to no-wrinkling. The dotted curves are yield critical curves for workpieces with various values of ratio h/b. When the values of $\sigma_a b^2 h/D$ fall below a dotted curve, the flange of the corresponding workpiece is in an elastic state, otherwise, it yields. These curves are obtained by taking $E/Y = 500$. It is seen from curve Γ_2 that for a given material with $E/Y = 500$ and $\nu = 0.3$, workpieces with $h/b = 0.025$, 0.020, etc., can yield without trouble from elastic wrinkling, but for that with $h/b = 0.015$ this expected process can be guaranteed only if $0 < \xi \leqslant \zeta^*$ or $\zeta^{**} \leqslant \xi < 1$. Thus, it is concluded that the smaller is h/b, the more easily elastic wrinkling occurs. Figure 5 should be very useful for engineers in designing deep-drawing processes.

Γ_2 is above Γ_1 and the distance between them is relative large. This shows that support of the die increases the critical stress. Hence it is expected that Γ_2 will be raised again if a blank-holder is employed [see Fig. 2(a)]. Moreover, Figs 3 and 4 indicate that the support of the die makes the number of waves increase. This appearance becomes more evident in a deep-drawing process with a blank-holder and the conclusion is in agreement with that of Johnson and Mellor [12].

A comparison is made in Fig. 5 between Γ_2 obtained by the present analysis and Γ_3 by the energy method [5]. It should be noted that Γ_2 is always below Γ_3. It may also be observed that the increasing critical stresses showed by Γ_3, when $\xi < 0.3$, is not reasonable. Our present results are more reasonable and more precise, and the method is more convenient.

Finally, we suggest the following two approaches to improve approximate solutions still further,

(1) Let $f_0(\rho) = c_0\Phi_0 + c_1\Phi_1 + \cdots + c_m\Phi_m$, where $\Phi_i (i = 0, 1, \ldots, m)$ satisfy certain conditions [9]. Then the Galerkin equation will yield

$$\int_{a/b}^{1} [L_1 f_0 + \xi_1 (L_2 f_0)] \Phi_i \, d\rho = 0 \quad (i = 0, 1, \ldots, m).$$

A more accurate solution than before can now be obtained by taking the determinant of the above equations to be equal to zero.

(2) The extended Kantorovich method [13] can be employed to find a better wrinkling mode.

Acknowledgements—The authors are grateful to Prof. W. Johnson for his helpful discussions concerning the present investigation.

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APPENDIX

(1) *Exact solutions of equation* (8)

These can be obtained for the following two special cases:

(i) $b \to \infty$

In this case we find,

$$f_0^{(4)} + \frac{2}{r} f_0''' + \frac{\zeta - 2n^2 - 1}{r^2}\left(f_0'' - \frac{1}{r} f_0'\right) + \frac{n^2(\zeta + n^2 - 4)}{r^4} f_0 = 0,$$

where

$$\zeta = -\sigma_a a^2 h/D.$$

This is an Eulerian equation which has a solution of the form

$$f_0 = \rho_*[A_1 \sin(\omega_1 \ln\rho_*) + A_2 \cos(\omega_2 \ln\rho_*) + A_3 sh(\omega_1 \ln\rho_*) + A_4 ch(\omega_2 \ln\rho_*)],$$

in which $\rho_* = r/a$,

$$\omega_1 = \sqrt{\tfrac{1}{2}\zeta - (1+n^2) + \sqrt{2n^2(2-\zeta) + \tfrac{1}{4}\zeta^2}}$$

and

$$\omega_2 = \sqrt{-\tfrac{1}{2}\zeta + (1+n^2) + \sqrt{2n^2(2-\zeta) - \tfrac{1}{4}\zeta^2}}.$$

(ii) $n = 0$

The buckling mode in this case is axisymmetric. Hence we get

$$f_0^{(4)} + \frac{2}{r} f_0''' + \left(\frac{\xi_1 - 1}{r^2} - \frac{\xi_1}{b^2}\right) f_0'' - \frac{1}{r}\left(\frac{\xi_1 - 1}{r^2} - \frac{\xi_1}{b^2}\right) f_0' = 0.$$

The equation can be integrated once and then the expression $f_0'/(1-\xi_1)$ replaced by F. This procedure gives

$$F'' + \frac{1}{r} F' + \left(\lambda^2 - \frac{1}{r^2}\right) F = \frac{c}{r}, \tag{A1}$$

where

$$\lambda^2 = -\xi_1/[(1-\xi_1)b^2]$$

and c is a constant. The homogeneous equation of equation (A1) is a Bessel equation. Hence it can easily be solved.

(2) *The explanation of inequality* $f_0(\rho) \geq 0$

According to (13), equation $f_0(\rho) = 0$ yields

$$\cot[\omega \ln(a/b)] = \cot[\omega \ln(r/b)].$$

The roots of this equation are

$$r = a\,\mathrm{e}^{2k\pi/[4\nu n^2-(1-\nu)^2]^{\frac{1}{2}}}, \quad (k = 0, 1, 2, \ldots).$$

Obviously, $k = 0$ makes $r = a$. The requirement of $f_0(\rho) \geqslant 0$ is equivalent to

$$a\,\mathrm{e}^{2\pi/[4\nu n^2-(1-\nu)^2]^{\frac{1}{2}}} \geqslant b.$$

This inequality is satisfied throughout the present calculations.