Solving mechanics problems with physical understanding

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This paper introduces a simple method for solving solid mechanics problems. Based on a physical consideration, the equilibrium equations of a mechanics system were transferred to corresponding dynamic ones. The static solution was then obtained as a result of damped oscillation. The bending of a simply supported elastic beam was used to demonstrate the feasibility and efficiency of the method. The paper emphasizes that solid mechanics problems can be solved effectively with a sound physical understanding.

NOMENCLATURE

$\mathbf{C}$	damping matrix, see equations (3) to (5)
c	system damping factor, defined by equation (5)
$\mathbf{F}$	vector of generalized external forces, see equations (1) and (2)
$\mathbf{M}$	mass matrix, see equations (3) and (4)
N	total node number of a discrete system
N_{max}	maximum iteration number with respect to fictitious time, defined by equation (7)
$\boldsymbol{P}$	vector of internal forces of the discrete system, see equations (1) and (2)
$\boldsymbol{R}$	vector of residual forces, defined by equation (3)
u, v, w	displacement components of a continuum, see equation (2)
$\boldsymbol{X}$	vector of generalized solution defined by equation (1)
$\dot{\boldsymbol{X}}, \ddot{\boldsymbol{X}}$	vectors of fictitious velocity and acceleration, respectively, defined by equation (3)
τ	increment of fictitious time

Superscripts

0	initial
n	number of iteration with respect to fictitious time

Subscripts

i, j	indices for vector or matrix elements

1. INTRODUCTION

The methods for solving solid mechanics problems in undergraduate teaching in mechanical engineering are usually mathematical. This is well demonstrated by most textbooks. Thus, in students' eyes, solutions to the governing equations are mainly the application of dry mathematical algorithms. The physical background of practical mechanics systems appears unrelated to the methods of solution. In this way, students' interest can hardly be stimulated.

This paper aims to introduce an alternative approach for solving the mathematical governing equations in solid mechanics. The approach is intended to make full use of physical arguments. In so doing, students may not need to carry heavy mathematical equipment, and may therefore enjoy the modelling of solid mechanics problems whose equations are difficult to solve.

2. SOLUTION WITH PHYSICAL UNDERSTANDING

2.1 An intuitive example

An unconventional idea was proposed physically by Rayleigh in the nineteenth century, as indicated by Timoshenko [1]: the static solution of a mechanics system could be viewed as the steady-state part of a corresponding dynamic problem. This idea can be interpreted by a simple mechanics model composed of a rigid ball and an elastic spring in a container with liquid, as shown in Fig. 1. The initial equilibrium position of the ball is at *O*. If the ball is subjected to a horizontal impact induced by any disturbance, it will vibrate. Because of the damping of the liquid, however, the oscillation will become smaller and smaller and the ball will finally stop at its initial equilibrium position *O*. Clearly, the dynamic solution of this ball–spring system is obtained when the equation of motion of the ball is solved. In the meantime, the static solution of the system is also obtained when the oscillation stops. The major difference in obtaining the dynamic and static solutions lies in that the detailed transient process of the system is not of interest when seeking the static solution by studying the dynamic process. What is important is to obtain the static solution in the shortest computation time [2].

The above idea can be easily generalized and implemented for solving general mechanics problems.

2.2 Solution strategy for a general problem in solid mechanics

Assume that with the aid of the finite difference or finite element method, a three-dimensional continuum in static equilibrium has been transferred into an equivalent discrete system connected through N nodes. The discrete governing equations of this static system can then be written as

$$P(X^{\Delta}) = F \tag{1}$$

where P is the vector of discrete internal forces in the continuum†, X^{Δ} is the vector of the real solution of the discrete system and F is the vector of the discrete external forces applied. If the node displacements (u_i, v_i, w_i) $(i = 1,\ldots, N)$ are the only numbers of node freedom of the discrete system, P, X^{Δ} and F are the vectors with $3N$ elements, i.e.

† For a linear system, $P(X^{\Delta})$ becomes KX^{Δ} where K is the stiffness matrix.

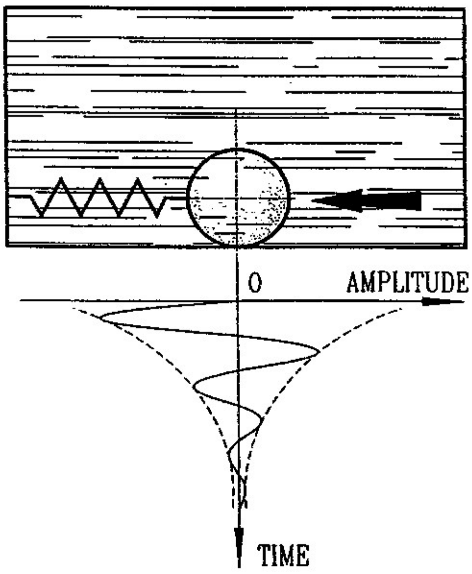


Fig. 1. The oscillation of a rigid ball.

$$
\begin{aligned}
X^{\Delta} &= \left\{x_1^{\Delta},\ldots,x_i^{\Delta},\ldots,x_N^{\Delta}\right\}^{\mathrm{T}}, &\quad x_i^{\Delta} &= \left\{u_i, v_i, w_i\right\} \\
P &= \left\{p_1,\ldots,p_i,\ldots,p_N\right\}^{\mathrm{T}}, &\quad p_i &= \left\{p_i^u, p_i^v, p_i^w\right\} \\
F &= \left\{f_1,\ldots,f_i,\ldots,f_N\right\}^{\mathrm{T}}, &\quad f_i &= \left\{f_i^u, f_i^v, f_i^w\right\}
\end{aligned}
\tag{2}
$$

However, when the real solution of equation (1), X^{Δ}, is replaced by an approximate solution X, residual forces $R = F - P(X)$ appear and cause disequilibrium of the system. Similarly to the oscillation of the above ball–spring system, now one can naturally consider that it is the disequilibrium forces R that induce the motion of the originally static system. Hence, the dynamic equations corresponding to equation (1) must be

$$\mathbf{M}\ddot{X} + \mathbf{C}\dot{X} = R \tag{3}$$

where M and C are mass and damping matrices and $\dot{X}$ and $\ddot{X}$ are velocity and acceleration vectors.

As mentioned before, because the motion of the system is not of practical interest, equation (3) can be referred to as the representation of a fictitious dynamic process. This means that one can fictitiously construct equation (3) in a way that simplifies the implementation of the method to the full extent. For this reason, one may consider **M** and **C** as fictitious mass and damping, take them as diagonal matrices and further assume that $\mathbf{C} = c\mathbf{M}$. That is,

$$\mathbf{M}=\begin{pmatrix}\mathbf{m}_{11} & \mathbf{0} & \dots & \mathbf{0}\\ \mathbf{0} & \mathbf{m}_{22} & \dots & \mathbf{0}\\ \dots & \dots & \dots & \dots\\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{m}_{NN}\end{pmatrix}, \qquad \mathbf{m}_{ii}=\begin{pmatrix} m_{ii}^{u} & 0 & 0\\ 0 & m_{ii}^{v} & 0\\ 0 & 0 & m_{ii}^{w}\end{pmatrix} \tag{4}$$

$$\mathbf{C}=\begin{pmatrix}c\mathbf{m}_{11} & \mathbf{0} & \dots & \mathbf{0}\\ \mathbf{0} & c\mathbf{m}_{22} & \dots & \mathbf{0}\\ \dots & \dots & \dots & \dots\\ \mathbf{0} & \mathbf{0} & \dots & c\mathbf{m}_{NN}\end{pmatrix}=c\begin{pmatrix}\mathbf{m}_{11} & \mathbf{0} & \dots & \mathbf{0}\\ \mathbf{0} & \mathbf{m}_{22} & \dots & \mathbf{0}\\ \dots & \dots & \dots & \dots\\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{m}_{NN}\end{pmatrix} \tag{5}$$

where c is a constant for all the node freedom and is called the system damping factor.‡ In so doing, explicit iteration formulae can be achieved when the central finite difference scheme with respect to fictitious time is applied, that is,

$$\begin{cases}\dot{X}^{n+1/2}=\dfrac{2-\tau^n c}{2+\tau^n c}\dot{X}^{n-1/2}+\dfrac{2\tau^n}{2+\tau^n c}\mathbf{M}^{-1}R^n\\ X^{n+1}=X^n+\tau^{n+1}\dot{X}^{n+1/2}\end{cases} \tag{6}$$

where superscript n indicates the nth iteration step and τ is the increment of fictitious time. In the derivation of equation (6), it has assumed that the external force vector, F, is independent of fictitious time and that

$$\begin{cases}\dot{X}^{n-1/2}=\left(X^n-X^{n-1}\right)/\tau^n\\ \ddot{X}^n=\left(\dot{X}^{n+1/2}-\dot{X}^{n-1/2}\right)/\tau^n\\ \dot{X}^n=\frac{1}{2}\left(\dot{X}^{n-1/2}+\dot{X}^{n+1/2}\right)\end{cases}$$

An assembly of equations (1) to (6) leads to a simple iterative algorithm as follows, where N_{max} is the maximum preset number of iterations:

(a) *specify* N_{max}; *let* $X^0=\mathbf{0}$, $n=0$
(b) *compute/guess* X^0 *and* c; *form* $\mathbf{M}$
(c) *calculate disequilibrium force* R^n
(d) *if* $R^n\approx\mathbf{0}$, *stop; otherwise continue*
(e) *calculate* $X^{n+1/2}$ *using equation (6a)*
(f) *determine* X^{n+1} *by equation (6b)*
(g) *apply boundary conditions*
(h) $n=n+1$; *if* $n=N_{max}$, *stop; otherwise return to step (c)*

The above algorithm includes three key factors: (i) the determination of fictitious mass matrix $\mathbf{M}$, (ii) the calculation of the damping factor c, and (iii) the selection of the initial

‡ It is possible to introduce different factors of instant critical damping for individual node freedom, see Refs [2–4].

vector X^0. The criterion of selecting M, c and X^0 is that the static solution of equation (3) is achieved in a minimum number of iteration steps. To obtain a high numerical stability, M should be calculated using the Gerschgorin theorem of eigenvalue disks (see Refs [2–5] for details). However, c and X^0 can be determined with a sound physical consideration.

2.3 Determination of the damping factor c

From the physical point of view of vibration, the ball–spring system illustrated in Fig. 1 reaches its static equilibrium in the shortest time when the liquid in the container provides a critical damping. For a general mechanics system represented by equation (3), an instant critical damping factor at the nth step of iteration, c^n, should be applied. According to Rayleigh's quotient, c^n can be calculated by

$$c^n = 2\left\{\frac{(X^n)^{\mathrm{T}} P(X^n)}{(X^n)^{\mathrm{T}} \mathbf{M}^n X^n}\right\}^{1/2} \tag{8}$$

2.4 Determination of the initial vector X^0

If an iteration starts from an initial vector close to the real solution, computation time can also be reduced. A good X^0 can be obtained by the following straightforward consideration.

The total energy of an undamped elastic system in vibration is a constant. This indicates that any initially guessed vector $X^{(1)}$ must correspond to an extreme state of X. Hence, if another extreme state of X, say $X^{(2)}$, is obtained by monitoring the undamped vibration,

$$X^0 = \tfrac{1}{2}\left(X^{(1)} + X^{(2)}\right) \tag{9}$$

is a very good initial vector for the subsequent calculation with instant critical damping. Fig. 2 schematically demonstrates this process of initial vector determination.

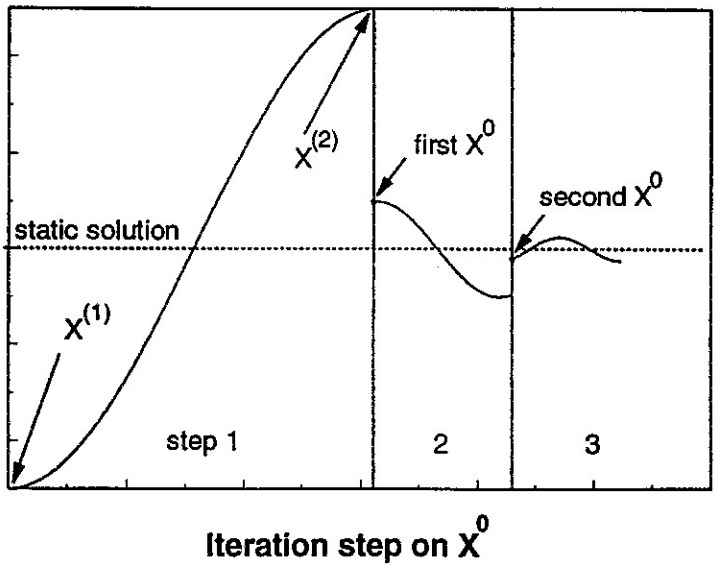
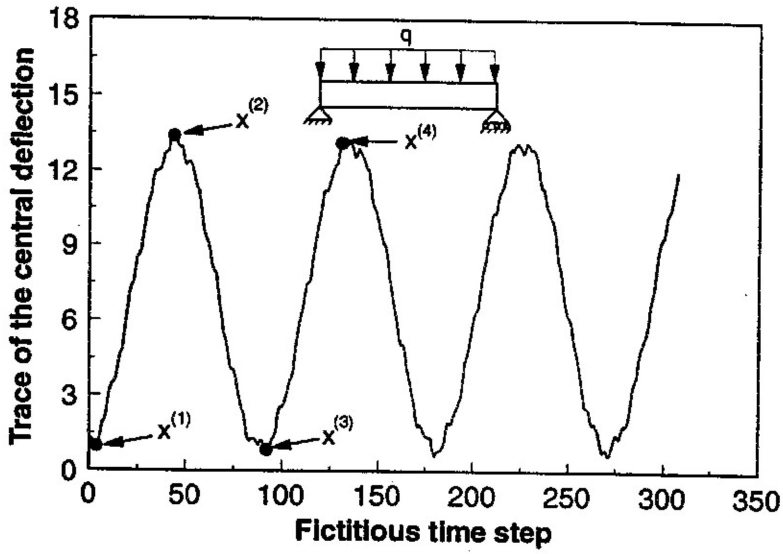
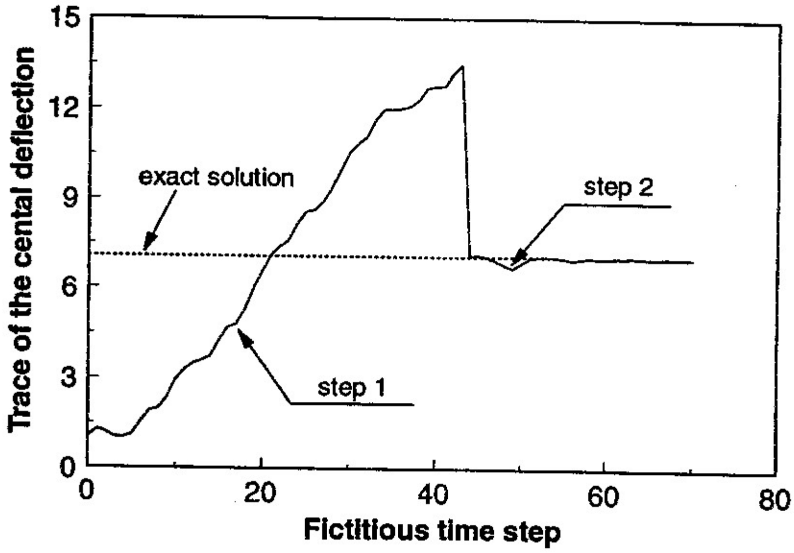


Fig. 2. Initial vector determination.



(a)



(b)

Fig. 3. Deflection of a simply supported beam. (a) Trace of central deflection during free vibration; (b) comparison of the central deflection with the exact solution.

3. ILLUSTRATION

As an illustrative example, let us consider the elastic bending of a simply supported beam subjected to a uniform pressure q. The cross-section of the beam is square. The geometrical and material parameters of the beam are $qh/E = 0.017\,52$ and $h/l = 0.052\,56$, where E is Young's modulus, h is the height and l is the length of the beam. In the numerical calculation, the beam was divided into eight finite difference segments. The free vibration trace of the beam deflection is shown in Fig. 3(a). Using algorithm (7) in conjunction with the initial vector and instant critical damping factor determined by equations (8) and (9), the static solution can be obtained in sixty fictitious time steps (see Fig. 3(b)). It should be pointed out that the technique discussed above can also be used to solve more complex engineering problems with both geometrical and material nonlinearity.

4. CONCLUDING REMARKS

This paper describes an approach of solving solid mechanics problems with physical consideration. It is an interesting experiment to demonstrate to students that sound physical understanding is important to improve the efficiency of a solution method. Sometimes even an intuitive idea can create a new technique for solving mathematical equations.

REFERENCES

[1] Timoshenko, S. P., *History of Strength of Materials*, McGraw-Hill, New York, 1953.
[2] Zhang, L. C., Kadkhodayan, M. and Mai, Y.-W., 'Development of the maDR method', *Computers & Structures*, 52, 1–8 (1994).
[3] Kadkhodayan, M. and Zhang, L. C., 'A consistent DXDR method for elastic-plastic problems', *International Journal for Numerical Methods in Engineering*, 38, 2413–2431 (1995).
[4] Yu, T. X. and Zhang, L. C., *Plastic Bending: Theory and Applications*, World Scientific, Singapore, 1995.
[5] Watson, W. A., Philipson, T. and Oates, P. J., *Numerical Analysis: The Mathematics of Computing*, 2nd edn, Edward Arnold, London, 1981.