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DEVELOPMENT OF THE maDR METHOD

L. C. ZHANG, M. KADKHODAYAN and Y.-W. MAI

Centre for Advanced Materials Technology, Department of Mechanical and Mechatronic Engineering, The University of Sydney, NSW 2006, Australia

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Abstract—This paper addresses the development of the modified adaptive dynamic relaxation (maDR) method. Through a critical review of the merits and disadvantages of previous studies, the primary physical approaches for improving the efficiency of the method are discussed. Emphases are placed on the selection of initial vectors and the calculation of fictitious damping factors. A variety of engineering problems, linear and non-linear as well as isotropic and anisotropic problems, are chosen to demonstrate the applicability and efficiency of the modified method. It is concluded that selection of initial vectors is only suitable for one type of engineering problems. However, a proper evaluation of fictitious damping and its combination with initial vector selection form a more general way to improve the efficiency of the maDR method.

NOTATION

$\mathbf{C}$	damping matrix, see eqns (3), (5) and (6)
c	system damping factor, defined by eqn (6)
$\mathbf{F}$	vector of generalized external forces of the discrete system, see eqns (1) and (2)
$\mathbf{K}$	stiffness matrix defined by eqn (9)
$\bar{\mathbf{K}}$	diagonal stiffness matrix defined by eqn (13)
$\mathbf{M}$	mass matrix, see eqns (3) and (5)
N	total node number of a discrete system
$N_{\max}$	maximum iteration number with respect to fictitious time, defined by eqn (8)
$\mathbf{P}$	vector of internal forces of the discrete system, see eqns (1) and (2)
$\mathbf{R}$	vector of residual forces, defined by eqns (3) and (4)
u, v, w	displacement components of a continuum, see equation (2)
$\mathbf{X}$	vector of generalized solution defined by eqn (1)
$\dot{\mathbf{X}}, \ddot{\mathbf{X}}$	vectors of fictitious velocity and acceleration, respectively, defined by eqn (3)
ζ	critical node damping factor, defined by eqn (21)
τ	increment of fictitious time

Superscripts

L	indication of three perpendicular directions coincident with displacement directions u, v, w, respectively
n	number of iteration with respect to fictitious time

Subscripts

i, j	indices for vector or matrix elements

1. DYNAMIC RELAXATION METHOD: A BRIEF OVERVIEW

Seeking efficient and stable numerical methods is an important part in solving complex engineering problems by means of computational mechanics. In the application of finite difference and finite element methods, the basic problems in dealing with strong non-linearities like those associated with large plastic deformation are the rate of convergence and the stability of numerical methods for solving the resultant large-scale discrete equations. However, most commonly used methods, like the Newton-type methods, cannot meet all the needs in engineering analysis.

An unconventional idea was proposed physically by Rayleigh (in [1]) in the nineteenth century: the static solution of a mechanics system could be viewed as the steady-state part of a corresponding dynamic problem. The successful application of this idea was first achieved in the 1960s through a set of linear elastic structural problems by Otter [2] and Day [3], and was named the dynamic relaxation (DR) method. Through the efforts of numerous researchers in the last two decades [4–31], the DR method has shown its broad applicability in solving various problems that conventional methods cannot.

1.1. *Overall characteristics of the method*

There are two ways to understand the DR method. Physically, as Rayleigh's original idea suggested, the transient process of the corresponding dynamic system of a static problem is not of special interest. The importance is to obtain the static solution in the shortest computation time and to guarantee that iterations are seeking the true solution robustly. Mathematically, the DR method can also be generated from the second-order Richardson method. Its convergence acceleration could then be investigated in a pure mathematical way [27, 28]. Although the family of the DR algorithms has both implicit and explicit formulations, the latter has received much attention in the application of computational solid mechanics.

Specifically, the characteristics of the explicit DR method are represented by its distinctive features as follows:

(a) It is incredibly reliable and stable in seeking equilibrium solutions even for extremely strong non-linear problems that cannot be solved in any other way [21–27, 31]. Moreover, the method needs no special techniques in studying bifurcation problems and it can incorporate easily with the dynamic criterion for stability (that is the only correct stability criterion for non-conservative problems [25]) which is otherwise very difficult to apply with other methods. The DR method therefore provides great potential for deformation and stress analyses as well as solving the buckling and micro-buckling problems of various components with different material characteristics; and

(b) It has a fixed simple algorithm such that, like any other commonly used methods, a standard computer solver can be programmed. Furthermore, because of its explicit formulation, only small data storage space is required. The method therefore offers broad flexibility in using different types of computers, large or micro-computers, which is of particular importance to most industrial researchers.

1.2. *Usual algorithm of DR method with explicit formulation and the existing problems*

1.2.1. *Usual algorithm.* Assume that with the aid of finite difference or finite element method, a three-dimensional continuum subjected to a set of external forces has been transferred into an equivalent discrete system connected through N nodes. The discrete governing equations of this static system can then be written as

$$\mathbf{P}(\mathbf{X}^{\Delta}) = \mathbf{F}, \tag{1}$$

where $\mathbf{P}$ is the vector of discrete internal forces, $\mathbf{X}^{\Delta}$ the vector of real solution of the discrete system and $\mathbf{F}$ is the vector of discrete external forces. If the node displacements (u_i, v_i, w_i) $(i = 1, \ldots, N)$ are the only numbers of node freedom of the discrete system, $\mathbf{P}$, $\mathbf{X}^{\Delta}$ and $\mathbf{F}$ are the vectors with $3N$ elements, i.e.

$$\begin{aligned}
&\mathbf{X}^{\Delta} = \{\mathbf{x}_1^{\Delta}, \ldots, \mathbf{x}_i^{\Delta}, \ldots, \mathbf{x}_N^{\Delta}\}^T \\
&\mathbf{P} = \{\mathbf{p}_1, \ldots, \mathbf{p}_i, \ldots, \mathbf{p}_N\}^T \\
&\mathbf{F} = \{\mathbf{f}_1, \ldots, \mathbf{f}_i, \ldots, \mathbf{f}_N\}^T \\
&\mathbf{x}_i^{\Delta} = \{u_i, v_i, w_i\}, \\
&\mathbf{p}_i = \{p_i^u, p_i^v, p_i^w\} \\
&\mathbf{f}_i = \{f_i^u, f_i^v, f_i^w\}.
\end{aligned} \tag{2}$$

When the real solution of (1), $\mathbf{X}^{\Delta}$, is replaced by an approximate solution $\mathbf{X}$, residual forces $\mathbf{R} = \mathbf{F} - \mathbf{P}(\mathbf{X})$ appear and in turn cause disequilibrium of the system. Hence, the corresponding dynamic equations are given by

$$\mathbf{M}\ddot{\mathbf{X}} + \mathbf{C}\dot{\mathbf{X}} = \mathbf{R}, \tag{3}$$

where $\mathbf{M}$ and $\mathbf{C}$ are mass and damping matrices and $\dot{\mathbf{X}}$ and $\ddot{\mathbf{X}}$ are velocity and acceleration vectors. Equation (3) states that the movement of the originally static system is induced by

$$\mathbf{R} = \{\mathbf{r}_1, \ldots, \mathbf{r}_i, \ldots, \mathbf{r}_N\}^T, \quad \mathbf{r}_i = \{r_i^u, r_i^v, r_i^w\}, \tag{4}$$

the disequilibrium forces. Clearly, as the real movement is not of interest, eqn (3) could be referred to as a fictitious dynamic process. This enables us to construct (3) in such a way that $\mathbf{M}$ and $\mathbf{C}$ are diagonal matrices, i.e.

$$\mathbf{M} = \begin{bmatrix} \mathbf{m}_{11} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{m}_{22} & \cdots & \mathbf{0} \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{m}_{NN} \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} \mathbf{c}_{11} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{c}_{22} & \cdots & \mathbf{0} \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{c}_{NN} \end{bmatrix}$$

$$\mathbf{m}_{ii} = \begin{bmatrix} m_{ii}^u & 0 & 0 \\ 0 & m_{ii}^v & 0 \\ 0 & 0 & m_{ii}^w \end{bmatrix}, \quad \mathbf{c}_{ii} = \begin{bmatrix} c_{ii}^u & 0 & 0 \\ 0 & c_{ii}^v & 0 \\ 0 & 0 & c_{ii}^w \end{bmatrix}, \tag{5}$$

and that $\mathbf{C} = c\mathbf{M}$, i.e.

$$\mathbf{C} = \begin{bmatrix} c\mathbf{m}_{11} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & c\mathbf{m}_{22} & \cdots & \mathbf{0} \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{0} & \mathbf{0} & \cdots & c\mathbf{m}_{NN} \end{bmatrix} = c \begin{bmatrix} \mathbf{m}_{11} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{m}_{22} & \cdots & \mathbf{0} \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{m}_{NN} \end{bmatrix} \tag{6}$$

which simplify the investigation of the method to the full extent. Explicit iteration formulae are achieved when the central finite difference scheme with respect to fictitious time is applied, that is

$$\dot{\mathbf{X}}^{n+1/2} = \frac{2-\tau^n c}{2+\tau^n c}\dot{\mathbf{X}}^{n-1/2} + \frac{2\tau^n}{2+\tau^n c}\mathbf{M}^{-1}\mathbf{R}^n$$

$$\mathbf{X}^{n+1} = \mathbf{X}^n + \tau^{n+1}\dot{\mathbf{X}}^{n+1/2}, \tag{7}$$

where superscript n indicates the nth iteration step and τ is the increment of fictitious time. An assembly of eqns (1)–(7) leads to the usual DR algorithm as follows, where $N_{\max}$ is the maximum preset iteration number:

(a) specify $N_{\max}$; let $\dot{\mathbf{X}} = \mathbf{0}$, $n = 0$
(b) compute/guess $\mathbf{X}^0$, c; form $\mathbf{M}$
(c) calculate disequilibrium force $\mathbf{R}^n$
(d) if $\mathbf{R}^n \approx \mathbf{0}$, stop; otherwise continue
(e) calculate $\dot{\mathbf{X}}^{n+1/2}$ by eqn (7a)
(f) determine $\mathbf{X}^{n+1}$ by eqn (7b)
(g) apply boundary conditions
(h) $n = n + 1$; if $n = N_{\max}$, stop; otherwise return to step (c). (8)

The above algorithm shows that there are three key factors which influence significantly the efficiency and stability of the method: (i) the determination of fictitious mass matrix $\mathbf{M}$, (ii) the calculation of damping factor c and (iii) the selection of initial vector $\mathbf{X}^0$. A proper criterion for evaluating a good DR algorithm is that the static solution of (3) is achieved in a minimum number of iteration steps with high numerical stability. As a result, in-depth investigations into these three factors become critical.

1.2.2. *On the evaluation of mass matrix M.* There are four methods which have been used in evaluating the element values of the fictitious mass matrix $\mathbf{M}$:

(a) take $\mathbf{M}$ as a unit matrix [32], i.e. $\mathbf{M} = \mathbf{I}$;
(b) select different values of m_{ii}^L [$L = u, v, w$; see eqns (2)–(5)] in different direction L through the analysis of individual problems [9];
(c) assume that the value of m_{ii}^L is proportional to the corresponding diagonal stiffness matrix of the system, k_{ii}^L, i.e. take $m_{ii}^L = \alpha k_{ii}^L$, where α is a proportional constant [13, 18]; the system stiffness matrix is given by

$$\mathbf{K} = \frac{\partial \mathbf{P}}{\partial \mathbf{X}} \tag{9}$$

(d) take [5, 19]

$$m_{ii}^L \geqslant \tfrac{1}{4}(\tau^n)^2 \sum_{j=1}^{N} |k_{ij}^L|. \tag{10}$$

Unfortunately, only (d) is a mathematically more reasonable method. In fact, to reduce the iteration time for obtaining the steady-state solution, a direct way is to enlarge the increment of fictitious time. However, it has been shown that the increase of τ^n in algorithm (8) is limited by the stability condition of iteration (see, e.g. [5]). Consequently, according to the Gerschgörin theory [33], inequality (10) must be satisfied to guarantee the iteration stability.

1.2.3. *On the calculation of damping factor* c. Proper selection of fictitious factor c can improve the convergence rate of the DR method. Five main approaches have been proposed in the published literature:

(a) calculate the damping factor by $c = 2\omega_0$, where ω_0 is the lowest circular frequency of the mechanics system under free vibration and is evaluated through the relation between ω_0 and the maximum kinetic energy of the system [4, 9];
(b) take $m_{ii}^L = \alpha k_{ii}^L$; if

$$\lambda = \frac{\|\mathbf{X}^{n+1} - \mathbf{X}^n\|}{\|\mathbf{X}^n - \mathbf{X}^{n-1}\|} \tag{11}$$

approaches a constant value during iteration, take $c = 2\omega_\lambda$, where ω_λ is the lowest circular frequency corresponding to the constant λ (see [18] for details);
(c) calculate the lowest circular frequency with the aid of Rayleigh quotient for linear problems, and then take the damping factor at the nth step of iteration as [5]

$$c^n = 2\left\{\frac{(\mathbf{X}^n)^T\bar{\mathbf{K}}^n\mathbf{X}^n}{(\mathbf{X}^n)^T\mathbf{M}^n\mathbf{X}^n}\right\}^{1/2}, \tag{12}$$

where $\bar{\mathbf{K}}^n$ is a diagonal matrix with elements

$$\bar{k}_{ii}^n = \frac{p_i^L(\mathbf{X}^n) - p_i^L(\mathbf{X}^{n-1})}{\tau^n \dot{x}_i^{L(n-1/2)}}; \tag{13}$$

(d) make use of the principle of Rayleigh quotient and assume that the instant critical damping factor of a non-linear system at the nth step of iteration can be expressed as [19]

$$c^n = \left\{\frac{(\mathbf{X}^n)^T\mathbf{P}(\mathbf{X}^n)}{(\mathbf{X}^n)^T\mathbf{M}^n\mathbf{X}^n}\right\}^{1/2}; \tag{14}$$

(e) take a guessed value of c and keep it unchanged thereafter throughout the whole process (e.g. [17, 34]).

From the physical point of view of vibration, a system reaches its static equilibrium in the shortest time when a critical damping is applied. Method (e) above is clearly poor because it uses a trial and error methodology. Method (a) is not realistic and for non-linear problems since the kinetic energy of a non-linear system is usually a function with multi-extrema. Determination of its maximum value is therefore very difficult. There are also several

question marks in using method (b), on the other hand. Firstly, it needs a guessed value of c before λ reaches a constant value. This wastes much computation time. Secondly, the method requires to store three vectors $\mathbf{X}^{n-1}$, $\mathbf{X}^n$ and $\mathbf{X}^{n+1}$ during iteration which occupy extra memory space. Thirdly, it ignores the fact that only instant critical damping is meaningful for a non-linear system. This method is therefore actually a two-stage damping method which guesses the first c for obtaining ω_λ and then keeps $c = 2\omega_\lambda$ unchanged until the static solution is reached. Hence, it must need much iteration time. Methods (c) and (d) are based on the same idea of Rayleigh quotient to evaluate instant critical damping of the system and therefore are more reasonable. However, if we compare the difference of data storage and computation requirements in the calculation of c^n, (d) is much better, simply because (c) needs extra storage of $\mathbf{P}(\mathbf{X}^{n-1})$ and $\dot{\mathbf{X}}^{n-1/2}$ and extra computation to form matrix $\bar{\mathbf{K}}^n$. Furthermore, practical application also shows that (d) is more efficient and stable than (c) [19, 23–26].

1.2.4. *On the selection of initial vector* X^0. If an iteration starts from an initial vector closer to the real solution, computation time will remarkably be reduced.

Alwar *et al.* [16] suggested that the decaying exponential envelope of the damped oscillations could be used to determine approximately the static solution of a beam. Although they thought that this method could offer solutions in one step (without iteration), it has to be referred to as an alternative for calculating initial vector $\mathbf{X}^0$ because of its unacceptable accuracy. As shown in Fig. 1, Alwar *et al.* calculated

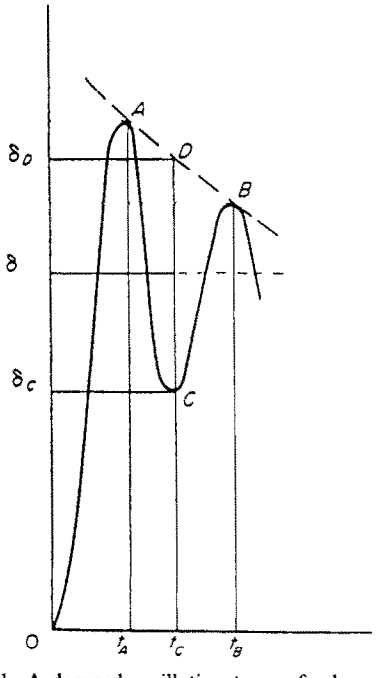

Fig. 1. A damped oscillation trace of a beam [16].

the approximate deflection of a beam, δ, by

$$\delta = \tfrac{1}{2}(\delta^D + \delta^C) \tag{15}$$

with an arbitrary value of damping factor c, where δ^D was evaluated by

$$\delta^D = C_1 \mathrm{e}^{-\alpha_1 t_C} \tag{16}$$

which is the rough approximation of the general expression of the decaying exponential envelope of the damped oscillations

$$\delta = C_0 + \sum_{i=1}^{N} C_i \mathrm{e}^{-\alpha_i t}, \tag{17}$$

where C_i and α_i are the system constants [16]. The method seems to be quite neat, but unfortunately it is not practical, because the oscillation trace used was a very special case. Most real structures vibrate in more complex ways, as illustrated in Fig. 2.

Another method for determining $\mathbf{X}^0$ was proposed by Zhang and Yu [19]. Before the calculation with critical damping factor, they monitored the first minimum and maximum points, x_i^{L*} and x_i^{L**} $(L = u, v, w; i = 1, \ldots, N)$, in the undamped oscillation. Then the elements of $\mathbf{X}^0$ were calculated by

$$x_i^{L0} = \tfrac{1}{2}(x_i^{L*} + x_i^{L**}). \tag{18}$$

This method is generally simple and can be easily applied. However, it also has two shortcomings. It needs to get two extreme points for each node freedom x_i^L, but as we can see later, x_i^{L**} is not necessary. Secondly, for general non-linear problems, the oscillation traces have complex profiles (e.g. see Fig. 2a), the efficiency of this method in solving these problems is then not remarkable.

It is the purpose of this paper to address further the development of the dynamic relaxation method from the physical point of view. According to the above discussions on the merits and disadvantages of previous studies, present emphases are placed on the selection of initial vectors and the calculation of fictitious damping factors. A variety of engineering problems, linear and non-linear, and isotropic and anisotropic problems, are chosen to demonstrate the applicability and efficiency of the modified methods. The effects of these factors in the application of the DR method to different engineering problems are successfully clarified. The paper finally points out that a proper calculation of fictitious damping combined with initial vector selection forms a general way to improve the efficiency of the maDR method.

2. FURTHER DEVELOPMENT OF THE maDR METHOD

A recent development of the DR method was made by Zhang and Yu [19]. They introduced the methods of calculating $\mathbf{X}^0$ by eqn (18) and c^n by eqn (14), and

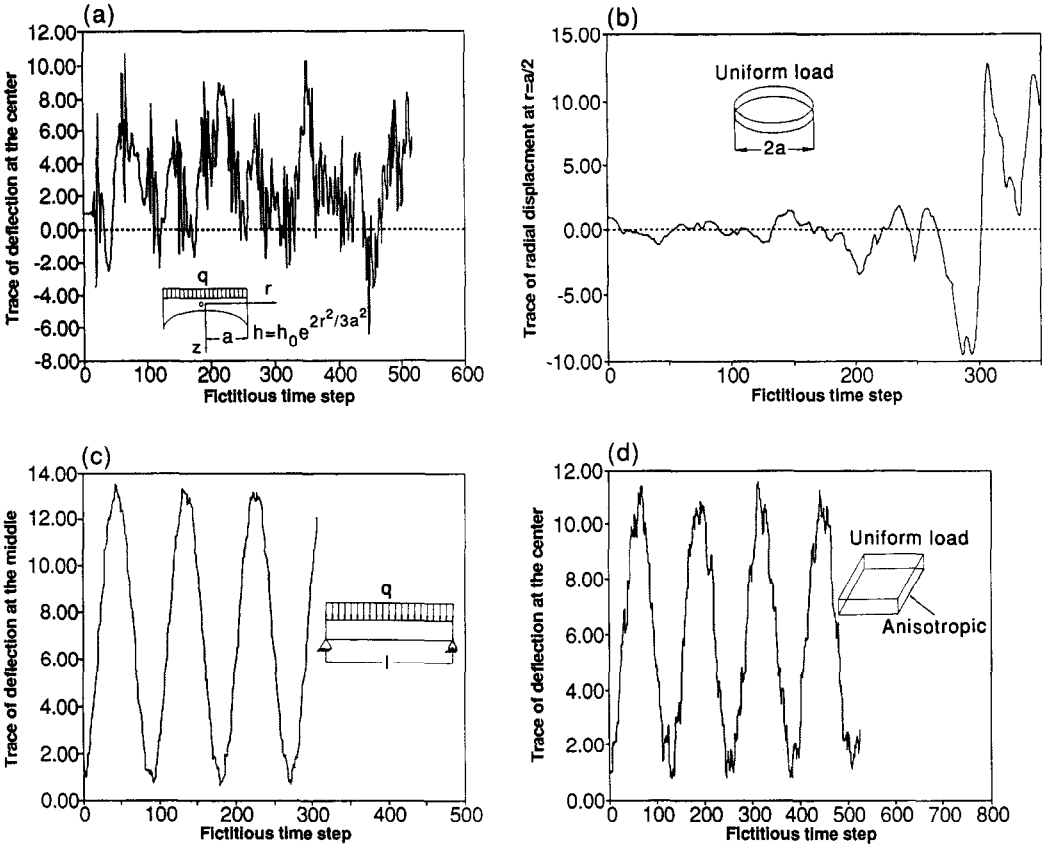

Fig. 2. The free vibration traces: (a) a circular plate with variable thickness, (b) a circular plate subjected to large deflection, (c) a simply supported beam, and (d) an anisotropic rectangular plate.

for the first time, combined these two factors together in one algorithm. An extra-convergence criterion, the kinetic energy criterion was also applied. This constructed a modified adaptive dynamic relaxation algorithm which have been shown to be more efficient than the others. However, the following improvement base on the maDR will provide far more powerful algorithms and help us to understand further the detailed effects of these factors.

2.1. *Improvement of the initial vector determination*

The total energy of an undamped system in vibration after the initial disturbance is a constant. This indicates that the initially guessed vector $\tilde{\mathbf{X}}_0$ must be an extreme state. Hence, the calculation of $\mathbf{X}^{**}$ in the maDR method is unnecessary and $\mathbf{X}^0$ can more conveniently be determined by

$$\mathbf{X}^0 = \tfrac{1}{2}(\tilde{\mathbf{X}}^0 + \mathbf{X}^*) \tag{19}$$

which saves half of the computation time compared with the application of (18).

In addition, an iteration scheme can provide much better $\mathbf{X}^0$ for problems with smooth oscillation traces, see Fig. 2(c). After the first calculation by (19), assign $\tilde{\mathbf{X}}^0 = \mathbf{X}^0$ and repeat the calculation again by (19) to obtain a new $\mathbf{X}^0$. Usually, two to three iterations will provide an accurate enough initial vector $\mathbf{X}^0$. Figure 3 presents schematically the convergence process.

2.2. *Application of critical node damping factor*

In all the DR algorithms discussed above, the whole discrete system has a common damping factor c such that eqn (7) holds. Although the maDR method evaluates instant c^n at each iteration step, it is not good enough. For a general non-linear system it is difficult to imagine that all nodes approach to their static equilibrium positions simultaneously when a unique damping is applied. This suggests that

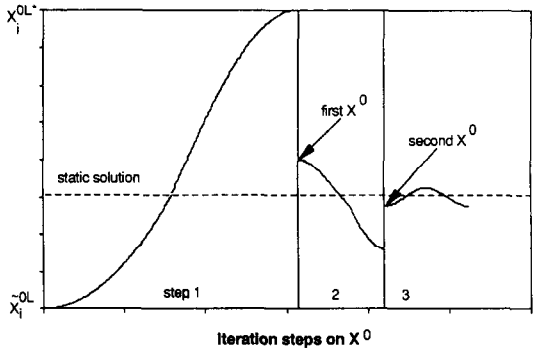

Fig. 3. A schematic diagram for the iteration of $\mathbf{X}^0$.

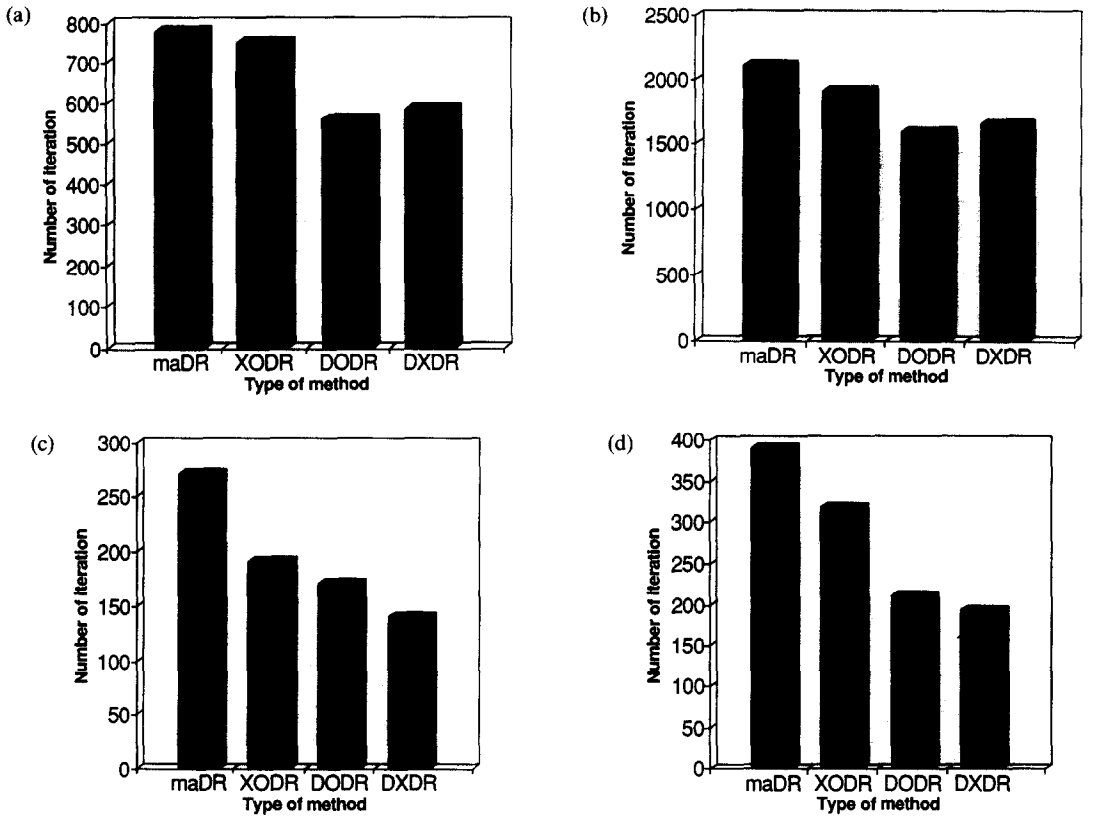

Fig. 4. A comparison of the efficiency between the maDR, XODR, DODR and DXDR methods; (a) the circular plate with variable thickness, (b) the circular plate subjected to large deflection, (c) the simply supported beam, and (d) the anisotropic rectangular plate.

different c values for different nodes should be introduced. We therefore define node damping factors ζ_i which satisfy

$$\mathbf{c}_{ii} = \zeta_i \mathbf{m}_{ii}, \quad i = 1, \ldots, N \tag{20}$$

to replace equation (6).† Accordingly, by applying Rayleigh's principle to each node we obtain the instant critical damping factor for node i at the nth iteration as

$$\zeta_i^n = 2\left\{\frac{(\mathbf{x}_i^n)^T \mathbf{p}_i^n}{(\mathbf{x}_i^n)^T \mathbf{m}_{ii}^n x_i^n}\right\}^{1/2}. \tag{21}$$

The explicit formulations similar to (7) can therefore be derived as

$$\dot{\mathbf{x}}_i^{n+1/2} = \frac{2 - \tau^n \zeta_i^n}{2 + \tau^n \zeta_i^n} \dot{\mathbf{x}}_i^{n-1/2} + \frac{2\tau^n}{2 + \tau^n \zeta_i^n} (\mathbf{m}_{ii}^n)^{-1} \mathbf{r}_i^n,$$

$$\mathbf{x}_i^{n+1} = \mathbf{x}_i^n + \tau^{n+1} \dot{\mathbf{x}}_i^{n+1/2}, \quad i = 1, \ldots, N. \tag{22}$$

† Different critical damping factor could also be introduced to different freedom on the same node if necessary. In this circumstance, ζ_i in eqn (20) should be replaced by a diagonal matrix of rank three

$$\zeta_i = \begin{pmatrix} \zeta_{ii}^u & 0 & 0 \\ 0 & \zeta_{ii}^v & 0 \\ 0 & 0 & \zeta_{ii}^w \end{pmatrix}, \quad i = 1, \ldots, N.$$

Combining the maDR algorithm [19] and the above improvements, we can propose a new DR algorithm as follows, where e_R and e_k are small positive constants specified by the user to define the convergence for disequilibrium forces and kinetic energy, respectively

(i) specify $N_{\max}$, e_R, e_k; let $\dot{\mathbf{X}} = \mathbf{0}$, $n = 0$,
(2) let $\zeta_i^0 = 0$ $(i = 1, \ldots, N)$; apply boundary conditions
(3) determine $\mathbf{X}^0$ by eqn (19); iterate $\mathbf{X}^0$ if necessary
(4) form $\mathbf{M}$

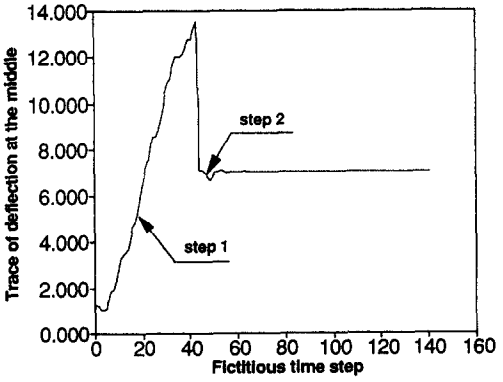

Fig. 5. The efficiency of the XODR method for the simply supported beam.

(5) calculate disequilibrium force $\mathbf{R}^n$
(6) if $|r_i^{L^n}| \leqslant e_R$, stop; otherwise continue
(7) calculate ζ_i^n by eqn (21)
(8) obtain $\dot{\mathbf{X}}^{n+1/2}$ by eqn (22a)
(9) if $\sum_{L=\mu}^{w} \sum_{i=1}^{N} (\dot{x}_i^{L^{n+1/2}})^2 \leqslant e_k$, stop; otherwise continue
(10) calculate $\mathbf{X}^{n+1}$ by eqn (22b)
(11) apply boundary conditions
(12) $n = n + 1$; if $n = N_{\max}$, stop; otherwise return to step (4). (23)

There are two alternatives in the algorithm (23), omission of steps (2) and (3) or of step (7). For the convenience of further discussion, we call the former DODR method as it takes into account damping factor only, and the latter XODR method because it is concerned with $\mathbf{X}^0$ only. The complete algorithm of (23) is naturally called DXDR method for its involvement with both damping and initial vector improvements.

3. ILLUSTRATION AND DISCUSSION

To illustrate the advantages of the improved algorithms and demonstrate the effects from different factors, a variety of problems, linear and non-linear as well as isotropic and anisotropic, are taken as computation examples. For the sake of succinctness, only typical results are presented.

According to their dynamic behaviour in the sense of dynamic relaxation (i.e. according to their fictitious dynamic behaviour when the DR method is concerned), engineering problems can be roughly divided into two types. Type (i) includes those problems with regularly periodic dynamic traces subjected to free oscillation, like those of a beam and rectangular isotropic and anisotropic plates shown in Figs 2(c, d). Type (ii) consists of the other problems with very irregular oscillation traces in undamped vibration, such as a circular plate in large deflection and a circular plate with variable thickness shown in Figs 2(a, b). (Note that all the plates and beams are simply supported and loaded uniformly with an intensity q.) The efficiency of maDR, DODR, XODR and DXDR are compared in Fig. 4. It is clear that the new improvements on the selection of the initial vector $\mathbf{X}^0$ and the introduction of instant node critical damping ζ_i are very successful. Overall, all the new methods are more efficient than the original maDR method.

It should be pointed out, however, that DODR, XODR and DXDR have different capabilities in solving different types of problems. XODR is very efficient for the problems of type (i) because regularly distributed smooth oscillation traces are ideal for the $\mathbf{X}^0$ iteration as shown in Fig. 5. DODR is generally suitable to both the problems of types (i) and (ii) but is less efficient than DXDR for the former. DXDR method is better than XODR and DODR for type (i) problems but worse than DODR for type (ii) problems simply because of the inefficiency of XODR in solving the problems of this type.

In summary, however, the DXDR method, i.e. the complete algorithm (23) may be recommended for solving all types of problems.

4. CONCLUSIONS

Through a critical review of previous studies on the explicit DR methods, new approaches to improve the efficiency of the algorithms of the DR family have been discussed. New algorithms called XODR, DODR and DXDR have also been proposed and tested for computational efficiency for a range of problems. It is shown that the DXDR method is a more general algorithm for all engineering problems. In particular, based on previous experience of using the maDR method to plasticity and bifurcation [22–26], the present new methods are most suitable for complex engineering applications such as elastic–plastic deformation studies and residual stress analysis.

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