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THE ELASTIC–PLASTIC PURE BENDING AND SPRINGBACK OF L-SHAPED BEAMS

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Abstract—L-shaped beams subjected to pure elastic–plastic bending are investigated thoroughly. The shifting and rotating of the neutral axis of the beam during bending and springback after unloading are described mathematically, while the curvatures of the beam before and after springback are given. The method proposed in this paper is valuable both to the further theoretical studies of the asymmetric elastic–plastic bending of beams with arbitrary cross-section and to its application in practical bending operations.

NOTATION

a	height or width of a L-shaped beam, as shown in Fig. 1
d	distance between neutral axis and a boundary line of the plastic region
E	Young's modulus
M	bending moment
m	dimensionless bending moment
p^*	distance from origin to the neutral axis, as shown in Fig. 1
t^*	thickness of the wing of the beam
x, y	co-ordinates, as shown in Fig. 1
α	an angle defined by Fig. 1
ε	axial strain
η	a parameter defined by equation (13e)
κ^*	curvature
ξ	a parameter defined by equation (11d)
σ	axial stress
σ_s	yield stress

1. INTRODUCTION

Over many decades the elastic–plastic bending and springback of beams has been one of the foci of investigations in Engineering Plasticity, since elastic–plastic bending is one of the most widely applied metal forming processes in industry. The symmetric bending of beams and their springback have been studied systematically and many useful results have been obtained [1–5]. However, very little attention has been paid to the case of asymmetric bending. Moreover, the researches on this subject have been restricted to the complete plastic bending of perfectly plastic beams [6–8].

Considering the importance of the asymmetric elastic–plastic bending of beams of various cross-section to the manufacturing engineering industry, the present paper aims to reveal the characteristics of the asymmetric bending and its springback and to search for a general method. To go further into this problem, L-shaped beams are investigated thoroughly. Final results show that this is a feasible approach.

2. ANALYSIS

The dimensions of an L-shaped beam and the direction of the vector of the bending moment loaded on it are shown in Fig. 1. The following assumptions are made in the analysis:

(i) the cross-section of the beam remains plane during bending;
(ii) only the stress normal to the cross-section is considered;
(iii) the material of the beam is elastic–perfectly plastic and isotropic;
(iv) the beam is stress-free before loading.

It is also assumed that no local unloading occurs during the monotonic increase of bending moment M. The validity of this will be verified later. In addition, as long as re-yielding does not occur during unloading, the unloading process can be considered as superposing an elastic effect produced by a bending moment equal to M in magnitude and opposite to it in loading direction.

In the Cartesian co-ordinate system shown in Fig. 1, the neutral axis of the beam can be described by equation

$$x \cos\alpha + y \sin\alpha - p^* = 0, \tag{1}$$

where α and p^* are as indicated in Fig. 1. Similarly, the boundary between elastic and plastic regions can be written as

$$x \cos\alpha + y \sin\alpha - p_i^* = 0, \tag{2}$$

where $i = 1$ or 2 indicates the boundary which is located at the upper or lower side of the neutral axis.

According to the assumption (i) and equation (1), compatibility of deformation requires

$$\varepsilon = \kappa^*(p^* - x\cos\alpha - y\sin\alpha), \tag{3}$$

where ε is the axial strain and κ^* is the absolute value of curvature. Hence, the constitutive equations can be expressed as follows when the assumptions (ii)–(iv) are noted

$$\begin{aligned} &\sigma = E\kappa^*(p^* - x\cos\alpha - y\sin\alpha), && \text{in the elastic region} \\ &\sigma = \pm\sigma_s, && \text{in the plastic region.} \end{aligned} \tag{4}$$

Moreover, the continuity of stress σ across the boundary between elastic and plastic regions gives

$$E\kappa^*(p_1^* - p^*) = E\kappa^*(p^* - p_2^*) = \sigma_s.$$

This can be rewritten as

$$E\kappa^* d^* = \sigma_s, \tag{5}$$

where the notation $d^* \equiv p_1^* - p^* \equiv p^* - p_2^*$ is adopted. Finally, the equilibrium of the beam requires

$$\iint \sigma \,\mathrm{d}x\mathrm{d}y = 0, \quad \iint \sigma x \,\mathrm{d}x\mathrm{d}y = 0 \quad \text{and} \quad \iint \sigma y \,\mathrm{d}x\mathrm{d}y = -M, \tag{6}$$

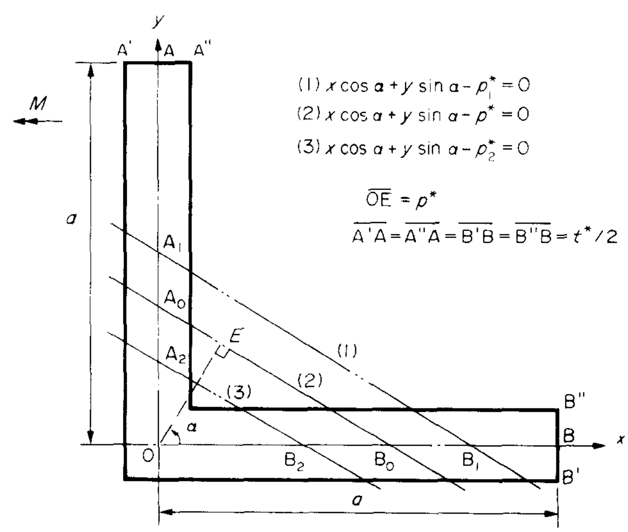

FIG. 1. A real L-shaped beam.

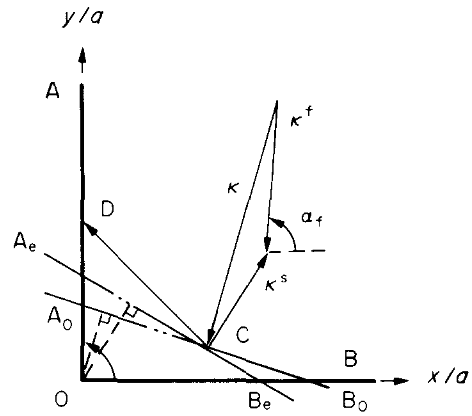

FIG. 2. An idealized L-shaped beam.

where the integrals are taken over the whole cross-section of the beam. Equations (4)–(6) are basic ones for the following analyses.

2.1. *The elastic–plastic bending of an idealized L-shaped beam*

In order to show the mechanism of the elastic–plastic bending of an L-shaped beam distinctly, an idealized model called an *idealized L-shaped beam* is first introduced. Such a model is seen in Figs 1 and 2, with the cross-section of the beam replaced by two straight lines of equal length, i.e. by $\overline{OA}$ and $\overline{OB}$. However, the area of an infinitesimal segment of length dl is regarded as t^*dl and the stress on this segment as σt^*dl, where t^* is the thickness of the original L-shaped beam. It can be expected that the idealized model will give a good approximation to the real beam when t^*/a is small.

The deformation process of the idealized L-shaped beam subjected to a pure bending moment can be divided into the following four stages.

(I) *Elastic bending stage.* The basic equations in this stage become

$$2p^*a-\tfrac{1}{2}a^2\cos\alpha-\tfrac{1}{2}a^2\sin\alpha=0, \tag{7a}$$

$$p^*=\tfrac{2}{3}a\cos\alpha, \tag{7b}$$

and

$$E\kappa^*(\tfrac{1}{2}p^*a^2-\tfrac{1}{3}a^3\sin\alpha)t^*=-M. \tag{7c}$$

These equations give

$$\alpha_e=\operatorname{arctg}\tfrac{5}{3}\doteq 59.036^\circ, \tag{8a}$$

and

$$p_e^*=2a/\sqrt{34}, \tag{8b}$$

where the subscript e denotes the elastic stage. Therefore, in the elastic stage, the neutral axis is fixed and described by $3x+5y-2a=0$.

A comparison of the distances from points A, B and O to the neutral axis indicates that point A will yield first. At this moment, the neutral axis rotates around the point $(\tfrac{2}{3}a, 0)$, and the maximum elastic bending moment and the maximum elastic curvature are

$$M_e=\tfrac{2}{9}\sigma_s a^2 t^*, \tag{9a}$$

and

$$\kappa_e^*=\sqrt{34}\,\sigma_s/(3aE), \tag{9b}$$

respectively. In the following analysis, dimensionless parameters used are

$$\kappa=\kappa^*/\kappa_e^*,\quad m=M/M_e,\quad t=t^*/a,\quad p=p^*/a,$$

and

$$p_i=p_i^*/a,\quad (i=1,2).$$

Obviously, stage I holds until $m=1$.

(II) *Elastic–plastic stage PI.* In this stage, segment $\overline{OB}$ is still in the elastic state, but the plastic region develops from point A towards O. The basic equations now become

$$p_1^2/(2\sin\alpha) - p_1 + 2p - \tfrac{1}{2}\cos\alpha = 0, \tag{10a}$$

$$p = \tfrac{2}{3}\cos\alpha, \tag{10b}$$

$$\kappa p_1^3 \sin^2\alpha = 18(\tfrac{1}{2} - 2m/9)/\sqrt{34}, \tag{10c}$$

and

$$\kappa(p_1 - p) = 3/\sqrt{34}. \tag{10d}$$

From these equations, we find

$$\alpha = \operatorname{arctg}\frac{5}{3(1-\xi^2)}, \tag{11a}$$

$$p_1 = (1-\xi)\sin\alpha, \tag{11b}$$

and

$$\kappa = \frac{3\sqrt{25+9(1-\xi^2)^2}}{\sqrt{34}(3-2\xi)(1-\xi)}, \tag{11c}$$

where

$$\xi = 1 - \tfrac{1}{15}[(9-4m) + 2\sqrt{(9-4m)(6-m)}]. \tag{11d}$$

This stage ends when $m = 45/32 \doteqdot 1.406$.

(III) *Elastic–plastic stage PII.* At this stage, point O is in a plastic state, but point B is still in the elastic range. The plastic region now develops along OB and extends towards point B. The basic equations change to

$$(p - d - \cos\alpha)^2 = 4pd\operatorname{ctg}\alpha, \tag{12a}$$

$$(p-d-\cos\alpha)^2(p-d+2\cos\alpha) = 3d\cos^2\alpha, \tag{12b}$$

$$(p^2 + d^2/3)/\sin^2\alpha = \tfrac{1}{2} - 2m/9 \tag{12c}$$

and

$$\kappa d = 3/\sqrt{34}. \tag{12d}$$

These lead to

$$\alpha = \operatorname{arctg}\left(-\frac{128\eta^3 - 192\eta^2 + 72\eta - 9}{64\eta^4}\right), \tag{13a}$$

$$p = \eta\sin\alpha, \tag{13b}$$

$$d = \eta\sin\alpha + 2\cos\alpha - 0.75\cos\alpha/\eta, \tag{13c}$$

and

$$\kappa = 3/(\sqrt{34}\,d), \tag{13d}$$

where η satisfies equations

$$\frac{9\eta(4\eta-1)^2}{8(\eta-1)(4\eta-1)^2 - 1} + \sqrt{3(0.5 - \tfrac{2}{9}m - \eta^2)} = 0 \tag{13e}$$

and

$$(3-\sqrt{3})/4 \leqslant \eta \leqslant 3/8. \tag{13f}$$

The expression (13) holds until $m = 9(6\sqrt{3} - 10)/2 \doteqdot 1.765$.

(IV) *Elastic–plastic stage PIII.* When $m \geqslant 9(6\sqrt{3}-10)/2$, point B will also yield. Thus we have

$$p\left(\frac{1}{\cos\alpha} + \frac{1}{\sin\alpha}\right) = 1, \tag{14a}$$

$$(p^2 + d^2/3)/\sin^2\alpha = \tfrac{1}{2} - 2m/9, \tag{14b}$$

$$(p^2 + d^2/3)/\cos^2\alpha = \tfrac{1}{2} \tag{14c}$$

and

$$\kappa d = 3/\sqrt{34}, \tag{14d}$$

and the corresponding solutions are

$$\alpha = \mathrm{arctg}\,[(1-4m/9)^{-1/2}], \tag{15a}$$

$$p = \sin\alpha\cos\alpha/(\sin\alpha+\cos\alpha), \tag{15b}$$

$$d = \sqrt{3(0.5\cos^2\alpha - p^2)} \tag{15c}$$

and

$$\kappa = 3/(\sqrt{34}\,d). \tag{15d}$$

For the fully plastic bending state, we have $\alpha_p = 67.5°$, $p \doteq 0.2706$ and $m \doteq 1.8640$.

Obviously, the above method can be easily applied to the analysis of various idealized L-shaped beams with different height and width.

2.2. *The analysis of the elastic–plastic bending of a real L-shaped beam*

It can be seen from the analysis of the idealized model that the deformation mechanism in every stage is very distinct and the method adopted above is preliminary and straightforward. It is then necessary to verify the practicability of the results from this model before the method is spread into wide use. To do so, a real L-shaped beam with equal height and width is considered (see Fig. 1). By performing a similar procedure as before, the following solutions can be obtained for each stage, respectively. In the following expressions, t denotes the relative thickness, i.e. $t = t^*/a$.

(I) *For the elastic stage.*

$$\alpha_e = \mathrm{arctg}\,\frac{5+7t^2/2-3t^4/16}{3-3t^2/2+3t^4/16}, \tag{16a}$$

$$p_e = (1-t^2/4)(\cos\alpha_e + \sin\alpha_e)/4, \tag{16b}$$

$$m_e = -9\left(\frac{1}{2}p_e - \frac{1}{3}\sin\alpha_e - \frac{t^2}{12}\sin\alpha_e - \frac{1}{8}p_e t^2\right)\Big/(2d_e) \tag{16c}$$

and

$$d_e = \sin\alpha_e + \tfrac{1}{2}t\cos\alpha_e - p_e. \tag{16d}$$

(II) *For the stage PI.*

$$p_1^2/(2\sin\alpha) - p_1 + 2p - \tfrac{1}{2}\cos\alpha = -t^2(\mathrm{ctg}\,\alpha\cos\alpha + 3\sin\alpha + 3\cos\alpha)/24, \tag{17a}$$

$$p = 2\cos\alpha/3 + (p_1 t^2\,\mathrm{ctg}\,\alpha)/6 + \tfrac{1}{4}pt^2, \tag{17b}$$

$$\kappa p_1^3/\sin^2\alpha = 18(\tfrac{1}{2} - 2m/9)/\sqrt{34} + 3\kappa p t^2/4 + \tfrac{1}{2}\kappa t^2\sin\alpha - \tfrac{1}{4}\kappa p_1 t^2\,\mathrm{ctg}^2\alpha \tag{17c}$$

and

$$\kappa(p_1 - p) = 3/\sqrt{34}. \tag{17d}$$

(III) *For the stage PII.*

$$(p-d-\cos\alpha)^2 = 4pd\,\mathrm{ctg}\,\alpha - t^2\sin^2\alpha/12, \tag{18a}$$

$$(p-d-\cos\alpha)^2(p-d+2\cos\alpha) = 3d\cos^2\alpha - dt^2(\mathrm{ctg}\,\alpha\cos^2\alpha + \tfrac{3}{4}\cos^2\alpha) - \tfrac{1}{4}(p-d)t^2\sin^2\alpha, \tag{18b}$$

$$\kappa d = 3/\sqrt{34} \tag{18c}$$

and

$$(p^2 + d^2/3)/\sin^2\alpha = \tfrac{1}{2} - 2m/9 - t^2\,\mathrm{tg}\,\alpha(p-d-\cos\alpha)/(12d) + t^2/8 - t^2\,\mathrm{ctg}^2\alpha/12. \tag{18d}$$

(IV) *For the stage PIII.*

$$(\cos^{-1}\alpha + \sin^{-1}\alpha)p = 1, \tag{19a}$$

$$(p^2+d^2/3)/\sin^2\alpha = \tfrac{1}{2} - 2m/9 + t^2\,\mathrm{tg}\alpha/6 + t^2/8 - t^2\,\mathrm{ctg}^2\alpha/12, \tag{19b}$$

$$(p^2+d^2/3)\cos^2\alpha = \tfrac{1}{2} + t^2\,\mathrm{ctg}\,\alpha/6 + t^2/8 - t^2\,\mathrm{tg}^2\alpha/12 \tag{19c}$$

and

$$\kappa d = 3/\sqrt{34}. \tag{19d}$$

The final results in the above three latter stages can be obtained with the aid of numerical methods.

2.3. *The springback analysis*

The attention of the present paper, as mentioned before, is restricted to the case of pure bending. Therefore, the axis of the beam always forms a plane curve, and the curvature κ can be regarded as a vector which is parallel to the principal normal of the curve. To show this clearly, the curvature vector is indicated by the boldface type $\boldsymbol{\kappa}$ later. Thus, the curvature vector of springback, $\boldsymbol{\kappa}^s$, is perpendicular to the elastic neutral axis, and has the magnitude

$$\kappa^s = m/m_e. \tag{20}$$

Let $\boldsymbol{\kappa}^f$ be the final curvature vector of the beam after springback. Then, from equation

$$\boldsymbol{\kappa}^f = \boldsymbol{\kappa} - \boldsymbol{\kappa}^s$$

with the help of equations (3) and (20), we can obtain the magnitude and the direction of vector $\boldsymbol{\kappa}^f$ as follows,

$$\kappa^f = \sqrt{\left[\kappa - \frac{m}{m_e}\cos(\alpha-\alpha_e)\right]^2 + \left[\frac{m}{m_e}\sin(\alpha-\alpha_e)\right]^2}, \tag{21a}$$

and

$$\alpha^f = \mathrm{arctg}\,\frac{\dfrac{m}{m_e}\sin(\alpha-\alpha_e)}{\kappa - \dfrac{m}{m_e}\cos(\alpha-\alpha_e)} + \alpha. \tag{21b}$$

The geometrical relationship of these vectors are as shown in Fig. 2.

3. DISCUSSION

Numerical calculation can be easily completed for every stage of deformation for the real L-shaped beam and its idealized model. These results are as shown in Figs 3–6 and Table 1. It can be seen that the most distinctive characteristic of asymmetric bending different from that of symmetric one is that the neutral axis of the former shifts and rotates simultaneously during elastic–plastic bending. This means that calculations of springback become very complicated. However, the idealized model proposed in the present paper can reduce these difficulties to a certain extent. This has been verified by the procedure of our analysis and the numerical calculation.

Table 1 indicates that the idealized model gives a good approximate solution to the real beam. For instance, in the case of $t = 0.1$, the maximum relative error of α, κ and p are 1.1, 10.0 and 3.0%, respectively. Comparison of the basic equations of the idealized model with those of the real beam shows that the former can be obtained from the latter when the terms containing t^2 and t^4 are all omitted. When the applied bending moment is such that the beam is deformed into stage PII or PIII, the results for the idealized model will be closer to the real solutions.

Figures 3 and 4 show that the rate of development of the plastic regions in the beam varies gradually. For the idealized model, this variation is monotonic, hence, no local unloading occurs during bending. However, local unloading will occur for a real L-shaped beam due to the rotation of the neutral axis during elastic–plastic bending. This can be ignored in applications to engineering problems when t is relatively small. Also, reverse yielding does not appear in the idealized beam.

In addition, Figs 3 and 4 show directly and quantitatively the relationship between the magnitude of bending moment m and the development of plastic zones in segments $\overline{\mathrm{OA}}$ and

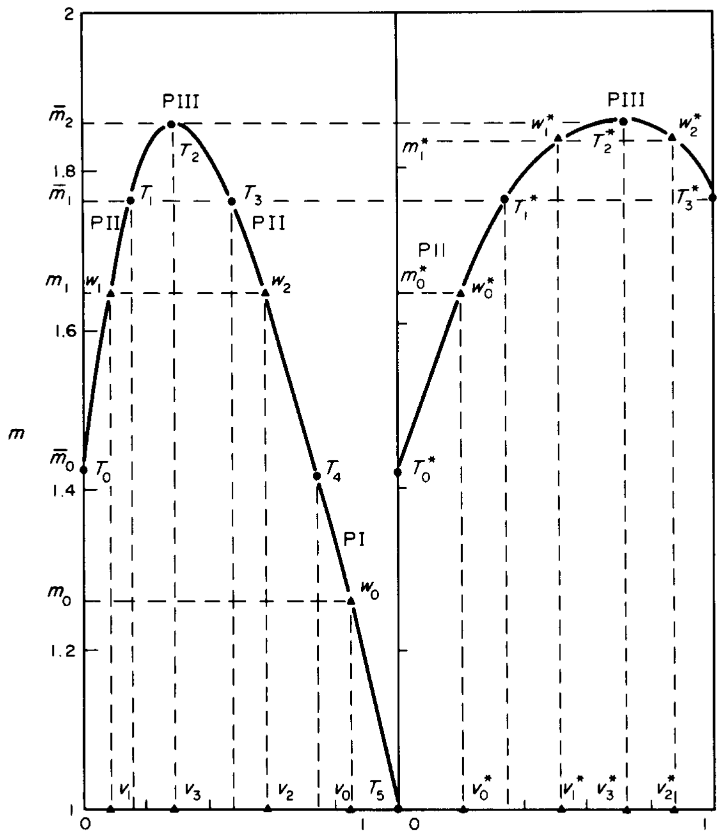

FIG. 3 FIG. 4

FIG. 3. Diagram of the development of the plastic region in segment $\overline{OA}$.

FIG. 4. Diagram of the development of the plastic region in segment $\overline{OB}$.

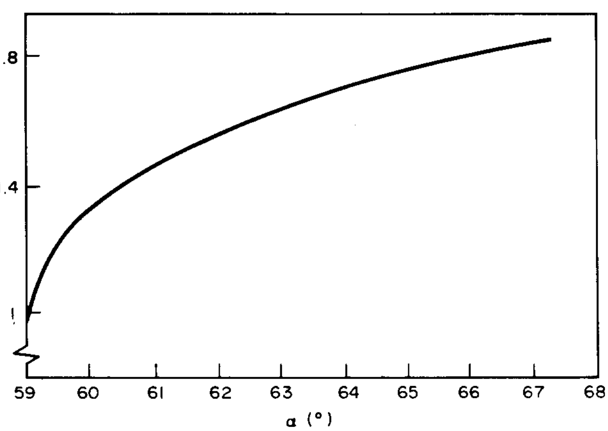

FIG. 5. Relationship of $m - \alpha$.

$\overline{OB}$ of the beam. These two figures reveal how the plastic zones appear, spread and finally join each other. It can be seen that the two segments of the beam, $\overline{OA}$ and $\overline{OB}$, undergo different processes of deformation. The former experiences three elastic–plastic deformation stages, i.e. PI, PII and PIII, but the latter experiences only two, that is PII and III. When $m = m_0$, for instance, segment $\overline{OB}$ is still in the elastic state, but a plastic zone over $\overline{v_0T_5}$ appears in segment $\overline{OA}$ at this time. As m increases, this plastic zone spreads gradually. When m exceeds

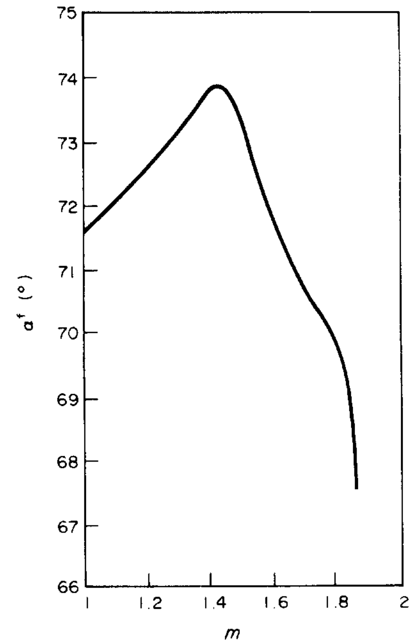

FIG. 6. Relationship of $m - \alpha^f$.

TABLE 1. COMPARISON OF REAL BEAMS WITH IDEALIZED ONES

Stage	m \ t	κ 0.0	κ 0.1	α (°) 0.0	α (°) 0.1	κ^f 0.0	κ^f 0.1	α^f (°) 0.0	α^f (°) 0.1
PI	1.000	1.108	1.081	59.128	59.403	0.008	0.018	72.043	63.300
	1.200	1.233	1.138	59.410	59.644	0.034	0.041	72.565	68.290
PII	1.635	2.374	2.269	62.924	62.816	0.751	0.698	71.412	70.701
	1.700	2.896	2.717	63.860	63.670	1.210	1.086	70.642	70.229
PIII	1.864	∞	—	67.500	—	∞	—	67.500	—
	1.882	—	∞	—	67.304	—	∞	—	67.334

$\bar{m}_0$, e.g. $m_1 (= m_0^*)$, two plastic zones will exist, $\overline{Ov_1}$ and $\overline{v_2T_5}$, in segment $\overline{OA}$, but only one, $\overline{Ov_0^*}$, in $\overline{OB}$. However, the two plastic zones in segment $\overline{OB}$ will appear when m exceeds $\bar{m}_1$ (e.g. m_1^*). The limiting case will be reached as m arrives at $\bar{m}_2$ and the two plastic zones of every segment will join each other at points v_3 for segment $\overline{OA}$ and v_3^* for $\overline{OB}$, respectively, at the same time. In these two figures, notation T_i ($i = 0, 1, \ldots, 5$) and T_i^* ($i = 0$, 1, 2, 3) are the characteristic points where the stage of deformation changes.

4. CONCLUSIONS

The investigation of the title problem performed in this paper is of great practical significance since elastic–plastic bending operation of metal components with asymmetric cross-section, such as angle steel, has been widely applied in manufacturing industry. Some new characteristics which cannot be obtained from the analysis of symmetric bending are revealed distinctly. These provide the basis for the further approach to the asymmetric elastic–plastic bending of the beams with arbitrary cross-section. In addition, the present results can be used to verify the correctness of those from other numerical procedures.

However, the analysis in the present paper is still preliminary. Many factors, such as the possibility of local buckling, reverse yielding, the Bauschinger effect and work hardening of the material, are not involved. These constitute subjects for further research in this field.

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